MODEL ORDER REDUCTION BASED ON MOMENT MATCHING USING LEGENDRE WAVELET AND HARMONY SEARCH ALGORITHM[*]

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Abstract– In this paper, a new method is investigated for model order reduction of high order systems based on moment matching technique. In this method, at first, full order model is expanded by Legendre wavelet function which is included in orthogonal functions. A suitable fixed structure model is considered as reduced order model whose parameters are unknown. These unknown parameters are determined using Harmony Search (HS) algorithm by minimizing the errors between the l first coefficients of Legendre wavelet expansion of full and reduced systems. The Routh criterion is applied for specifying the stability conditions. Therefore, the stability condition is considered as constraints in optimization problem. To present the ability of the proposed method, four test systems are introduced. The obtained results are compared with other conventional techniques such as Balanced Truncation (BT) method and Hankel Singular Value (HSV) method. The obtained results show the proposed approach performs very well with respect to other reduction methods.

Keywords– Model reduction, moment matching, legendre wavelet, harmony search algorithm, Routh array

1. INTRODUCTION

Various methods are reported in the literature for order reduction in time domain and frequency domain. Model reduction was started by Davison in 1966 [1] and followed by Chidambara by suggesting several modifications to Davison's approach [2-4]. After that different approaches were proposed such as: dominant pole retention [5], Routh approximation [6], Hurwitz polynomial approximation [7-8], stability equation method [9-10], moments matching [11-l4], continued fraction method [15-17], Pade approximation [18], etc.

The issue of optimality in model reduction was considered by Wilson [19-20] who suggested an optimization approach based on minimization of the integral squared impulse response error between the full and reduced-order models. This attempt was continued by other researches through other approaches [21-24].

In 1981 [25], the controllability and observability of the states was considered in model reduction by Moore. The suggested approach suffered from steady state errors but the stability of the reduced model was assured if the original system was also stable [26]. Furthermore, the concept of H_∞, H_2, L_2 and L_∞ was used in model reduction [27-30].

In the past decade, one of the most promising research fields has been "Evolutionary Techniques", an area utilizing analogies with nature or social systems [31-32]. The evolutionary techniques such as Particle Swarm Optimization (PSO) and Genetic Algorithm (GA) and Harmony Search (HS) algorithm are used for order reduction of systems [33-35]. In these approaches, the reduced order model's parameters are

achieved by minimizing a fitness function which is often Integral Square Error (ISE), Integral Absolute Error (IAE), H_2 norm or H_∞ norm [36-38].

In this paper a new alternative method is used for order reduction via Legendre wavelets function. Wavelet theory [39] has been applied in a wide range of engineering science; particularly, in signal analysis for waveform representation and segmentations, identification, optimal control and many other applications [40-43]. Wavelets permit the accurate representation of a variety of functions and operators.

In the proposed method, the full order system is expanded by Legendre wavelets function and then the *l* first coefficients of Legendre wavelets expansion are obtained. A desired fixed structure for reduced order model is considered and a set of parameters are defined whose values determine the reduced order system. These unknown parameters are determined using HS algorithm by minimizing the errors between the *l* first coefficients of Legendre wavelets expansion of full and reduced systems. To satisfy the stability, Routh criterion is applied as it is used in [35] where it is stated in optimization problem as constraints and subsequently, optimization problem is converted to a constrained optimization problem. Four test systems are reduced by the proposed method and compared with those available in the literature to show the accuracy of the proposed method.

To make a proper background, Legendre wavelets function and harmony search algorithm are briefly explained in Sections 2 and 3, respectively. The proposed method is explained in Section 4. The ability of the proposed approach is shown in Section 5. The paper is concluded in Section 6.

2. LEGENDRE WAVELETS

Wavelets have been successful in approximating the solution of different types of systems. They constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet $\psi(x)$.

Legendre wavelets $\psi_{n,m}(t) = \psi(\hat{n}, k, m, t)$ are defined as follows [44]:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m + \dfrac{1}{2}}\, 2^{\frac{k}{2}} P_m\left(2^k t - \hat{n}\right) & , \quad \dfrac{\hat{n}-1}{2^k} < t < \dfrac{\hat{n}+1}{2^k} \\ 0 & , \quad otherwise \end{cases} \tag{1}$$

$$m = 0,1,2,\ldots,M-1 \quad ,n = 1,2,\ldots,2^{k-1} \quad ,k = 0,1,2,\ldots \quad ,\hat{n} = 2n-1$$

where

$$\begin{aligned} P_0(t) &= 1 \\ P_1(t) &= t \\ P_{m+1}(t) &= \left(\frac{2m+1}{m+1}\right) t P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t) \quad m = 1,2,3,\ldots \end{aligned} \tag{2}$$

A function $f(t) \in L^2[0,1]$ can be approximated as:

$$f(t) = \sum_{m=0}^{M-1} \sum_{n=1}^{2^{k-1}} C_{n,m} \psi_{n,m}(t) \tag{3}$$

where $C_{n,m} = <f(t), \psi_{n,m}(t)>$, in which $<.,.>$ denotes the inner product as:

$$C_{n,m} = <f(t), \psi_{n,m}(t)> = \int_0^1 f(t), \psi_{n,m}(t) dt \tag{4}$$

Equation (3) can be written in a matrix form as:

$$f(t) = C^T \psi(t) \tag{5}$$

or

$$f(t) = \psi^T(t)C \tag{6}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices which are given by:

$$C^T = \begin{bmatrix} c_{1,0} & c_{1,1} & \cdots & c_{1,M-1} & c_{2,0} & \cdots & c_{2,M-1} & \cdots & c_{2^{k-1},0} & \cdots & c_{2^{k-1},M-1} \end{bmatrix}$$

$$\psi(t) = \begin{bmatrix} \psi(t)_{1,0} & \psi(t)_{1,1} & \cdots & \psi(t)_{1,M-1} & \psi(t)_{2,0} & \cdots & \psi(t)_{2,M-1} & \cdots & \psi(t)_{2^{k-1},0} & \cdots & \psi(t)_{2^{k-1},M-1} \end{bmatrix}^T$$

By determining the elements of matrices C and $\psi(t)$, the full order model can be expanded as a Legendre wavelet function.

The moment matching methods, the Krylov subspace methods [45], belong to the Projection based model order reduction (MOR) methods. These methods are applicable to non-parametric linear time invariant systems [46].

The transfer function is expanded by Legendre wavelets into a power series at an expansion point around s = 0. The moments of a function are defined as the coefficients of the Legendre wavelets expansion around a given point. The goal in moment-matching model reduction is the construction of a reduced order system where the moments of the reduced order model match the moments of the original system.

3. HARMONY SEARCH ALGORITHM

HS is based on natural musical performance, a process that searches for a perfect state of harmony. The harmony in music is analogous to the optimization solution vector, and the musician's improvisations are analogous to local and global search schemes in optimization techniques. The HS algorithm does not require initial values for the decision variables and uses a stochastic random search that is based on the harmony memory considering rate and the pitch adjusting rate.

In general, the HS algorithm works as follows [47- 48]:

Step 1. Initialization: Initial population is produced randomly within the range of the boundaries of the decision variables. The optimization problem can be defined as:

Minimize $f(x)$ subject to $x_{iL} < x_i < x_{iU}$ $(i = 1, 2, …, N)$ where x_{iL} and x_{iU} are the lower and upper bounds for decision variables.

The HS algorithm parameters are also specified in this step. They are the harmony memory size (*HMS*) or the number of solution vectors in harmony memory, harmony memory considering rate (*HMCR*), distance bandwidth (*bw*), pitch adjusting rate (*PAR*), and the number of improvisations (*K*), or stopping criterion. *K* is the same as the total number of function evaluations.

Step 2. Initialize the harmony memory (HM). The harmony memory is a memory location where all the solution vectors (sets of decision variables) are stored. The initial harmony memory is randomly generated in the region $[x_{iL}, x_{iU}]$ $(i = 1, 2, …, N)$. This is done based on the following equation:

$$x_i^j = x_{iL} + rand( ) \times (x_{iU} - x_{iL}) \quad j = 1, 2, \cdots, HMS \tag{7}$$

where $rand( )$ is a random from a uniform distribution of [0,1].

Step 3. Improvise a new harmony from the harmony memory. Generating a new harmony x_i^{new} is called improvisation where it is based on 3 rules: memory consideration, pitch adjustment and random selection. First of all, a uniform random number R is generated in the range [0, 1]. If R is less than HMCR, the decision variable x_i^{new} is generated by the memory consideration; otherwise, x_i^{new} is obtained by a random

selection. Then, each decision variable x_i^{new} will undergo a pitch adjustment with a probability of PAR if it is produced by the memory consideration. The pitch adjustment rule is given as follows:

$$x_i^{new} = x_i^{new} \pm R \times bw \tag{8}$$

Step 4. Update harmony memory. After a new harmony vector x^{new} is generated, the harmony memory will be updated. If the fitness of the improvised harmony vector $x^{new} = \left(x_1^{new}, x_2^{new}, \cdots, x_N^{new} \right)$ is better than that of the worst harmony, the worst harmony in the HM will be replaced with x^{new} and become a new member of the HM.

Step 5. Repeat steps 3 and 4 until the stopping criterion (maximum number of improvisations *K*) is met.

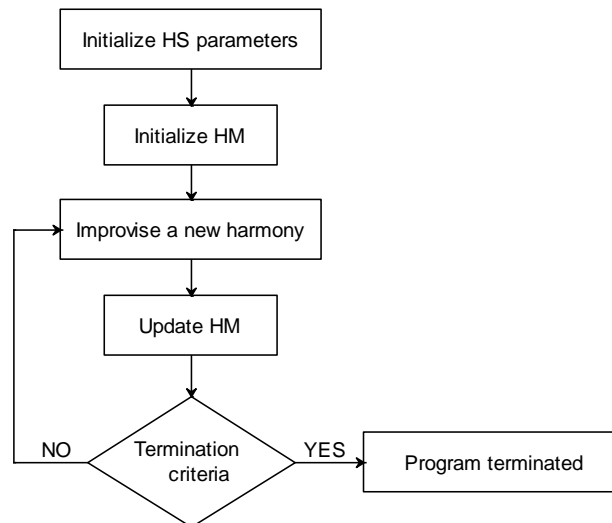The optimum design algorithm using HS is sketched basically as shown in Fig 1.

Fig. 1. Basic flowchart diagram for HS algorithm

4. THE PROPOSED MODEL REDUCTION METHOD

Consider an order *n* transfer function of SISO (stable single-input single-output) system as follows:

$$G(s) = \frac{a_1 s^{n-1} + a_2 s^{n-2} + ... + a_n}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + ... + b_n} \tag{9}$$

which a_i and b_i are constants.

The aim is to find a reduced model of order *r* in which *r* is smaller than *n* while the reduced order model has the principal and important specifications of the full order system. The reduced order system can be presented such as:

$$G_r(s) = \frac{c_1 s^{r-1} + c_2 s^{r-2} + ... + c_r}{s^r + d_1 s^{r-1} + d_2 s^{r-2} + ... + d_r} \tag{10}$$

in which $c_1, c_2, ..., c_r$ and $d_1, d_2, ..., d_r$ are unknown constants.

To obtain the reduced model by the proposed method, at first, we expand the full order system based on Legendre wavelets function. Then we obtain the *l* first coefficients of Legendre wavelet expansion of original system which is shown by $F_i \ i = 0,1,2,3,...,l$. Then a desired fixed structure is considered for reduced order model as defined in Eq. (10) in which $c_1, c_2, ..., c_r$ and $d_1, d_2, ..., d_r$ are unknown parameters of reduced order model that are found by HS. The purpose of the optimization is to obtain the best parameters for $G_r(s)$. Therefore, each harmony is a *d*-dimensional vector in which *d* is $c_r + d_r$. Each

harmony is a solution to G_r and for each solution (harmony), the Legendre wavelet expansion is obtained. Each harmony is evaluated by minimizing the following fitness function:

$$J^* = \sum_{i=0}^{l} \left(\left| F_i - \hat{F}_i \right| \right) \tag{11}$$

in which, $\hat{F}_i$ are the coefficients of Legendre wavelet expansion of reduced order system. The algorithm searches for the best harmony until the termination criteria are met. At this stage the best parameters are given as parameters of reduced order model.

Furthermore, the reduced model must be stable if the original system is stable. Therefore, the Routh criterion is applied to assure the stability. For specifying the stability conditions, following [35], the denominator polynomial of reduced order model in Eq. (10) can be shown as:

$$
\begin{aligned}
& s^r + h_1 s^{r-1} + (h_2 + h_3 + \cdots + h_r)s^{r-2} + h_1(h_3 + h_4 + \cdots + h_r)s^{r-3} + \\
& [h_2(h_4 + h_5 + \cdots + h_r) + h_3(h_5 + h_6 + \cdots + h_r) + \\
& h_4(h_6 + h_7 + \cdots + h_r) + \cdots h_{r-2}h_r]s^{r-4} + \cdots + h_{1+q}h_{3+q}\cdots h_{r-2}h_r
\end{aligned}
\tag{12}
$$

which is constructed by taking the coefficients of the first two rows of the Routh array whereby the elements of its first column have the following entries:

$$1, h_1, h_2, h_1h_3, h_2h_4, h_1h_3h_5, \ldots, h_{1+k}h_{3+k}\ldots h_{r-2}h_r \tag{13}$$

where, k is equal to 1 for even r and k is equal to 0 for odd r.

Comparing the entries of the array in Eq. (13) with $1, d_2, d_4, \ldots$ and those of the second row with $d_1, d_3, d_5, \ldots$ gives Eq. (14):

$$
\begin{aligned}
d_1 &= h_1 \\
d_2 &= (h_2 + h_3 + \ldots + h_r) \\
d_3 &= h_1(h_3 + h_4 + \ldots + h_r) \\
&\vdots \\
d_r &= (h_{1+k}h_{3+k}h_{r-2}h_r)
\end{aligned}
\tag{14}
$$

Substituting the above equations in reduced order model's denominator, Eq. (12) is achieved. Therefore, the necessary and sufficient condition for all roots of the reduced system to be strictly in the left-half plane is

$$
\begin{aligned}
h_1 &> 0 \\
h_2 &> 0 \\
&\vdots \\
h_r &> 0
\end{aligned}
\tag{15}
$$

And subsequently

$$
\begin{aligned}
d_1 &> 0 \\
d_2 &> 0 \\
&\vdots \\
d_r &> 0
\end{aligned}
\tag{16}
$$

Thus, to have an optimum stable reduced system, the reduced order model's parameters are determined by minimizing the following fitness function:

$$J = \sum_{i=0}^{l}\left(\left|F_i - \hat{F}_i\right|\right)$$

(17)

$$subject\ to\ d_j > 0 \quad for\quad j = 1, \cdots, r$$

Therefore, the reduced order model is achieved such that the *l* first coefficients of Legendre wavelet expansion of the full order system are equal with the *l* first coefficients of Legendre wavelet expansion of reduced order model.

The proposed method can be summarized in the following steps:

Step 1: Obtaining the Legendre wavelet functions of the full order system in Eq. (9).

Step 2: Considering a desire fixed structure for reduced order model as defined in Eq. (10), where $c_1, c_2, \ldots, c_r$ and $d_1, d_2, \ldots, d_r$ are unknown parameters of reduced order model that are found in the next step.

Step 3: Applying HS to find the unknown parameters. The goal of the optimization is to find the best parameters for $G_r(s)$. Therefore, each harmony is a *d*-dimensional vector in which *d* is $c_r + d_r$. Each harmony is a solution to G_r and for each solution (harmony), obtaining the Legendre wavelet functions. Evaluate each harmony using the objective function defined by equation (17) searching for the harmony associated with J_{best} until the termination criterion is met. At this stage the best parameters are given as parameters of reduced order model.

5. SIMULATION AND RESULTS

In this section, we show the efficiency and accuracy of the proposed method by applying it on four test systems. To obtain a reduced-order model, a step-by-step procedure is given for the first test system.

Test system 1: The system given in [49] by Singh is the first system to be reduced where a procedure is considered to obtain the reduced system. The system is as follows:

$$G(s) = \frac{18s^7 + 514s^6 + 5928s^5 + 36380s^4 + 122664s^3 + 222088s^2 + 185760s + 40320}{s^8 + 36s^7 + 546s^6 + 4536s^5 + 22449s^4 + 67284s^3 + 118124s^2 + 109584s + 40320}$$

(18)

By using Legendre wavelet function and HS, the reduced-order model can be achieved by the steps below:

Step 1: By choosing $t_f = 20$, $k = 2$ and $M = 4$, the Legendre wavelet expansion of the full order system in Eq. (18), based on section 2, can be written as:

$$\begin{cases} G = (4.3734)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{0.5})(Q_{10})\right) + (-0.6230)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{1.5})(Q_{11})\right) + \\ \quad (-0.1697)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{2.5})(Q_{12})\right) + (0.2261)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{3.5})(Q_{13})\right) + \cdots \quad 0 \le t \le \frac{t_f}{2} \\ \\ G = (2.5655)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{0.5})(Q_{20})\right) + (-0.3295)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{1.5})(Q_{21})\right) + \\ \quad (0.0372)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{2.5})(Q_{22})\right) + (-0.0040)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{3.5})(Q_{23})\right) + \cdots \quad \frac{t_f}{2} < t \le t_f \end{cases}$$

(19)

Step 2: The full order system in Eq. (18) is going to be reduced to a third-order system with the following transfer function:

$$G_r(s) = \frac{c_1 s + c_2}{s^2 + d_1 s + d_2} \quad (20)$$

where c_i and d_i are the unknown parameters of reduced order model.

Step 3: To obtain the unknown parameters HS is applied. Since, the aim of the optimization is to find the best parameters for $G_r(s)$, each harmony is a d-dimensional vector in which $d = 4$. The HMS is selected to be 4, HMCR and evaluation number is set to be 0.9 and 1000, respectively. Each harmony is a solution to G_r and for each solution (harmony), the Legendre wavelet expansion is obtained. Each harmony in the population is evaluated using the objective function defined by Eq. (17) searching for the best J until the termination criteria are met. At this stage the best parameters are given for reduced order model where the following reduced order model is obtained:

$$G_{Legendre\ Wavelet}(s) = \frac{17.4996s + 5.5457}{s^2 + 7.1578s + 5.5453} \quad (21)$$

The Legendre wavelet expansion of obtained reduced order model is as:

$$\begin{cases} G_r = (4.3735)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{0.5})(Q_{10})\right) + (-0.6230)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{1.5})(Q_{11})\right) + \\ \quad (-0.1739)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{2.5})(Q_{12})\right) + (0.2254)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{3.5})(Q_{13})\right) + \cdots \quad 0 < t \leq \frac{t_f}{2} \\ \\ G_r = (2.5506)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{0.5})(Q_{20})\right) + (-0.3329)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{1.5})(Q_{21})\right) + \\ \quad (0.0379)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{2.5})(Q_{22})\right) + (-0.0041)\left(2\left(\frac{1}{\sqrt{t_f}}\right)(\sqrt{3.5})(Q_{23})\right) + \cdots \quad \frac{t_f}{2} < t \leq t_f \end{cases} \quad (22)$$

Comparing Eq. (19) and Eq. (22) shows that a good approximant is achieved for $G(s)$.

The step response of the full order system and that of the system with second-order reduced models are shown in Fig. 2. This figure shows that, the reduced order model is an adequate low-order model that retains the characteristics of full order model. Also, to show the efficiency of the proposed method, the step and frequency responses of the obtained reduced model are compared with those available in the literature. Figures 3-4, show the comparison of the results obtained with the one proposed by Singh [49], Optimal Hankel norm approximation (HSV) [50] and Balanced Truncation (BT) [50], respectively.
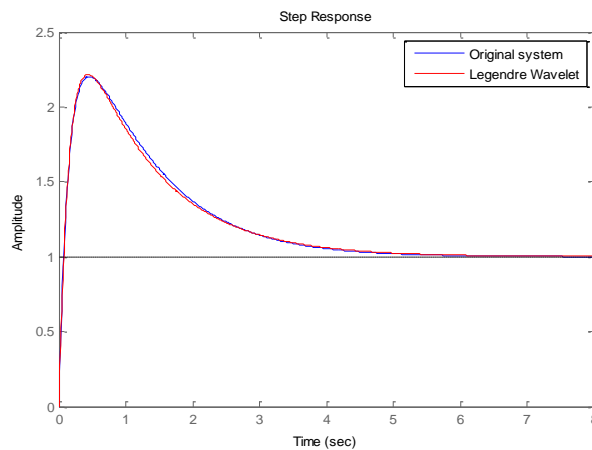

Fig. 2. Step response of full order and reduced order model by the proposed method for test system 1
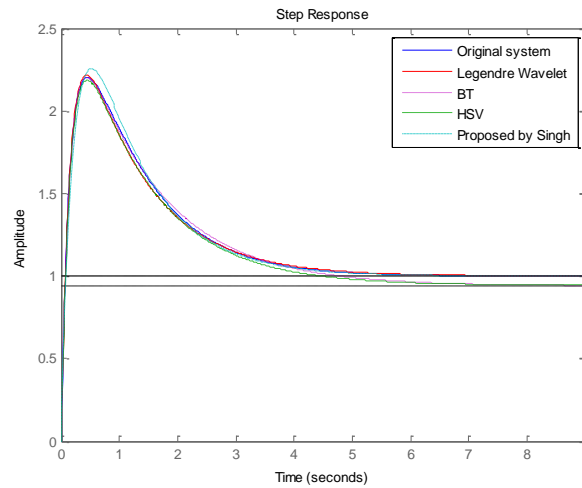
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Fig. 3. Step response of full order and reduced order model by the proposed method and
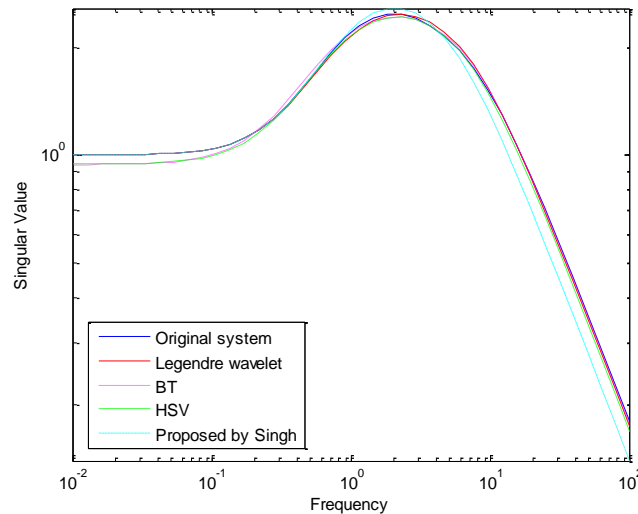other conventional methods for test system 1

Fig. 4. Frequency response of full order and reduced order model by the proposed
method and other conventional methods for test system 1

These figures show that the achieved results from the proposed method are very similar to original system compared to other methods, which will make the stability and performance characteristics of both systems be the same.

Furthermore, comparison between some specifications such as steady state value, rise time, settling time and maximum overshoot are given in Table 1. Also, ISE criterion between the full order and reduced order models ($e = |y - y_r|$) is given in Table 1. It is clearly seen that the specifications of reduced order model that is achieved by the proposed method are close to the specifications of original system.

Table 1. Comparision of methods for test system 1

	Steady state	Overshoot (%)	Rise time (sec)	Settling time (sec)	ISE
Original system	1	120	0.0569	4.82	-
Legendre wavelet	1	122	0.0577	5.03	0.0016
BT	0.94	134	0.0529	5.97	0.1200
HSV	0.944	132	0.0556	5.48	0.1109
Proposed by Singh	1	126	0.071	4.68	0.0101

Also, the plot of $e = |y - y_r|$ is illustrated in Fig. 5 for reduced systems. This figure shows that the obtained error by the proposed method in this paper is less than other methods.
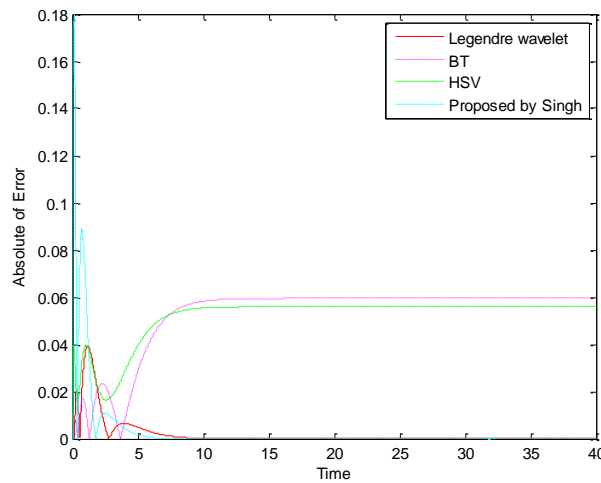

Fig. 5. The plot of $e = |y - y_r|$ for the full order and reduced systems by the
proposed method and other methods for test system 1

Test system 2: In [51], a procedure is presented to obtain a reduced order system by Mukherjee. To compare the proposed method with the proposed method by Mukherjee, the system given in [51] is adopted which is a ninth-order system:

$$G_{original\ system}(s) = \frac{s^4 + 35s^3 + 291s^2 + 1093s + 1700}{s^9 + 9s^8 + 66s^7 + 294s^6 + 1029s^5 + 2541s^4 + 4684s^3 + 5856s^2 + 4620s + 1700} \quad (23)$$

Based on the explanations given for test system 1, the obtained reduced system by the proposed method is as follows:

$$G_{Legendre\ Wavelet}(s) = \frac{0.4121s^2 - 2.9431s + 5.2356}{s^3 + 4.2602s^2 + 7.7705s + 5.2356} \quad (24)$$

The step response of the original system and the obtained reduced model are shown in Fig. 6. In this figure, the responses of the system with third-order primary reduced models obtained by other methods are also included for comparison. Also, the plot of $e = |y - y_r|$ is given in Fig. 7.
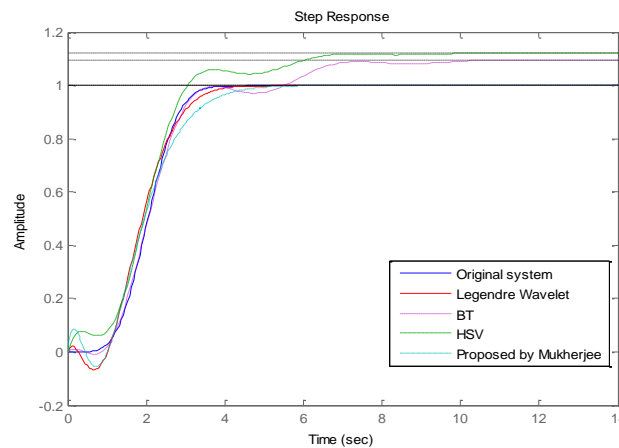

Fig. 6. Step response of full order and reduced order model by the proposed method
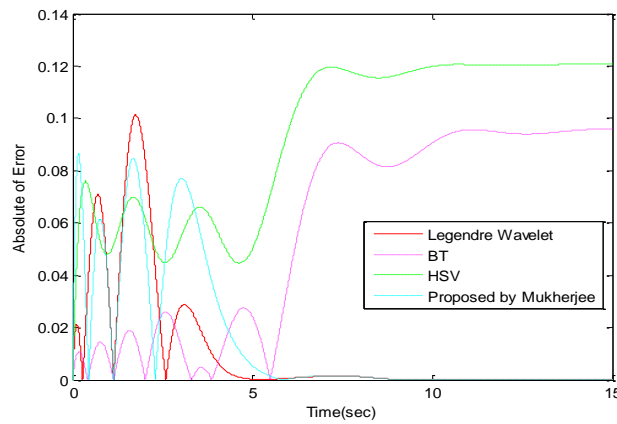and other conventional methods for test system 2

Fig. 7. The plot of $e = |y - y_r|$ for the full order and reduced systems by the
proposed method and other methods for test system 2

Furthermore, maximum overshoot, rise time, settling time, steady state value, ISE criterion are shown in Table 2. Once again, the results obtained confirm that a satisfactory approximation has been achieved. It is clearly seen that the specifications of reduced order model that is achieved by the proposed method are close to the specifications of original system and better than other methods.

Table 2. Comparision of methods for test system 2

	Steady state	Overshoot (%)	Rise time (sec)	Settling time (sec)	ISE
Original system	1	0	1.54	3.36	-
Legendre wavelet	1	0	1.73	3.69	0.0098
BT	1.1	0	2.11	6.62	0.0734
HSV	1.12	0	1.93	6.11	0.1479
Proposed by Mukherjee	1	0	2.02	4.32	0.0131

Test system 3: The third test system to be reduced is the following 9th order Boiler System [38]:

$$\dot{x}(t) = \begin{bmatrix} -0.910 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4.449 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.262 & 571.479 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -571.479 & -10.262 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.987 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15.214 & 11.622 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -11.622 & -15.214 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -89.874 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -502.665 \end{bmatrix} x(t)$$

$$+ \begin{bmatrix} -4.336 \\ -3.691 \\ 10.141 \\ -1.612 \\ 16.629 \\ -242.476 \\ -14.261 \\ 13.672 \\ 82.187 \end{bmatrix} u(t) \qquad (25)$$

$$y(t) = \begin{bmatrix} -0.422 & -0.736 & -0.00416 & 0.232 & -0.816 & -0.715 & 0.546 & -0.235 & -0.080 \end{bmatrix} x(t) \quad (26)$$

Based on the explanations given for test system 1, the obtained reduced system by the proposed method is as follows:

$$G_{Legendre\ Wavelet}(s) = \frac{151.34s^2 + 4388.22s + 4787.63}{s^3 + 31.96s^2 + 426.04s + 376.41} \quad (27)$$

The comparison of the proposed method with the one proposed by Salim in [38] and BT method and HSV method are shown in Figs. 8-9 and Table 3, which illustrate a better performance of the proposed method.

Test system 4: The last test system considered for model reduction is a 6th order discrete-time system [52] as follows:

$$G_{Original}(z) = \frac{0.3277z^6 + 0.9195z^5 + 1.038z^4 + 0.5962z^3 + 0.1618z^2 + 0.00698z - 0.005308}{z^6 + 1.129z^5 + 0.2889z^4 - 0.08251z^3 - 0.04444z^2 - 0.00476z} \quad (28)$$
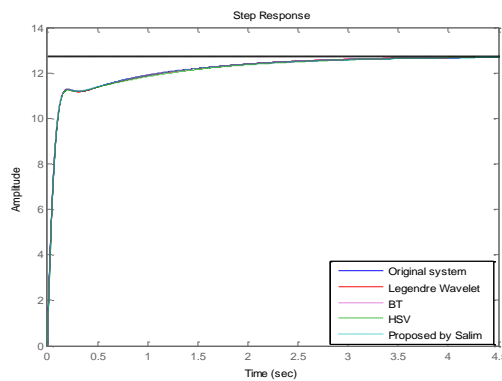

Fig. 8. Step response of full order and reduced order model by the proposed method and other conventional methods for test system 3
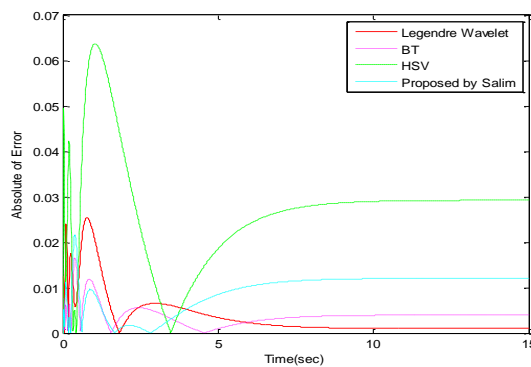

Fig. 9. The plot of $e = |y - y_r|$ for the full order and reduced systems by the proposed method and other methods for test system 3

Table 3. Comparision of methods for test system 3

	Steady state	Overshoot (%)	Rise time (sec)	Settling time (sec)	ISE
Original system	12.7	0	0.543	2.28	-
Legendre wavelet	12.7	0	0.557	2.26	6.32×10^{-4}
BT	12.7	0	0.539	2.24	0.0016
HSV	12.7	0	0.591	2.52	0.0859
Proposed by Salim	12.7	0	0.53	2.22	0.0139

In this paper, for model reduction of discrete systems, at first, the discrete time system is transformed to continuous time system by using bilinear Tustin transformation [53]. The sampling time of transformation is 1s.

$$G_{Tustin}(s) = \frac{s^5 + 15.6s^4 + 124.2s^3 + 510.3s^2 + 1166s + 959.3}{s^6 + 21s^5 + 175s^4 + 735s^3 + 1624s^2 + 1764s + 720} \tag{29}$$

Then, the continuous time system is reduced by the proposed method. Based on the explanations given for test system 1, the obtained reduced system by the proposed method is as follows:

$$G_{Legendre\ wavelet} = \frac{s^2 + 6.0008s + 19.2131}{s^3 + 9.4999s^2 + 22.2610s + 14.4177} \tag{30}$$

Figures 10-11 and Table 4 are the comparison with the proposed method and the one proposed by Narain [52], BT and HSV methods. The results show the accuracy and better performance of the proposed method.
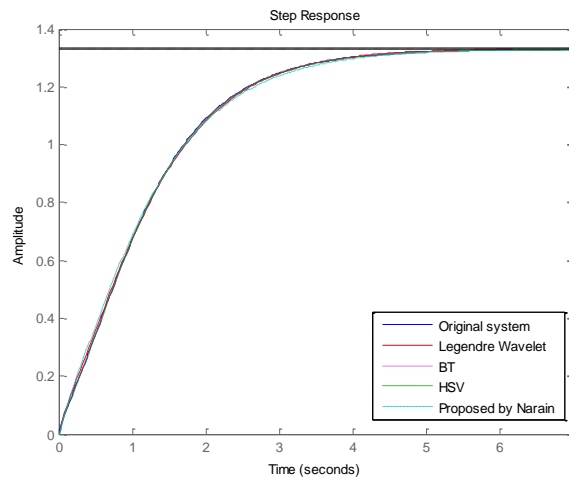

Fig. 10. Step response of full order and reduced order model by the proposed
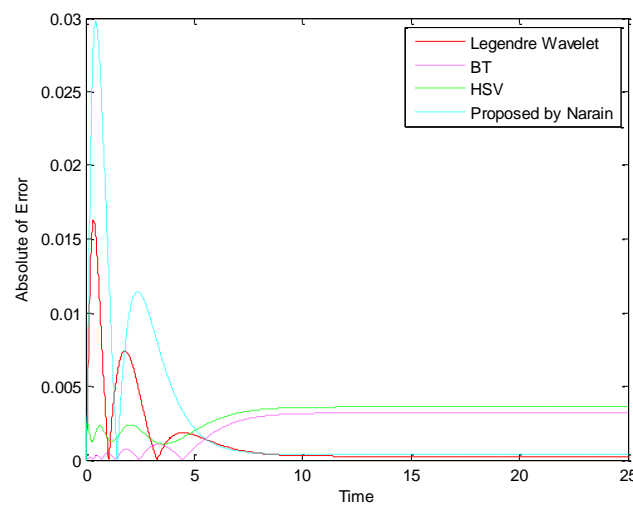method and other conventional methods for test system 4

Fig. 11. The plot of $e = |y - y_r|$ for the full order and reduced systems by the proposed
method and other methods for test system 4

Table 4. Comparision of methods for test system 4

	Steady state	Overshoot (%)	Rise time (sec)	Settling time (sec)	ISE
Original system	1.33	0	2.39	4.13	-
Legendre wavelet	1	0	2.44	4.07	1.80×10^{-4}
BT	1.1	0	2.37	4	1.87×10^{-4}
HSV	1.12	0	2.38	4.04	2.58×10^{-4}
Proposed by Narain	1	0	2.5	4.27	7.58×10^{-4}

Similarly transforming the continuous-time reduced model to the discrete-time reduced model using the bilinear Tustin transformation and taking sampling time 1s can be written as:

$$G_{Discrete\ Legendre\ wavelet} = \frac{0.3356z^3 + 0.6255z^2 + 0.3968z + 0.1068}{z^3 + 0.2456z^2 - 0.1455z - 0.0009} \tag{31}$$

Figure 12 shows step response of full order (original) discrete-time system and the reduced order discrete-time system based on the proposed method and other methods.
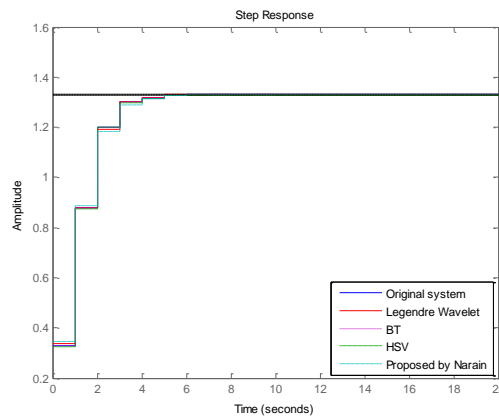

Fig. 12. Step response of full order of discrete-time model and reduced order of discete-time model by the proposed method and other conventional methods for test system 4

6. CONCLUSION

In this paper, an approach based on Legendre wavelet expansion for order reduction is investigated. To present the accuracy and efficiency of the method, four test systems are reduced by the proposed method. The proposed method is compared with some conventional order reduction techniques where the obtained results show that the proposed approach has high accuracy with respect to conventional order reduction methods.

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