Iranian Journal of Science & Technology, Vol. 23, No. 1, Transaction B
Printed in Islamic Republic of Iran, 1999


# TOTAL STABILITY OF FUZZY FEEDBACK CONTROL SYSTEMS*

V. TAHANI** AND F. SHEIKHOLESLAM

Department of Electrical and Computer Engineering, Isfahan University of Technology, Isfahan, I. R. of Iran

**Abstract:** In this paper, the total stability of specific fuzzy control systems is studied. The plant of the system is assumed to be a linear system and the fuzzy controller is of Sugeno type. The stability analysis is based on some newly developed theorems that guarantee sufficient conditions for the total stability of the system. The analysis results are applied to the stabilization problem of an unstable plant.

**Keywords** – Fuzzy control systems, asymptotical stability, total stability

## 1. INTRODUCTION

In the past, approaches to the analysis and design of fuzzy control systems have been extensively studied [1,2] and practical applications of such systems have been well demonstrated [2-4]. Most of the presented results on the analysis and design techniques of fuzzy systems are of experimental or heuristic type [1]. Recently, some analytical tools for stability analysis and design have been presented [5-8]. The presented works are concerned with the stability of equilibrium points of the systems. However, total stability is one of the most important properties for linear systems. A linear system is considered totally stable if and only if for any bounded input and any bounded initial state, the output and the state of the system are bounded [9]. This concept has been adopted for fuzzy systems as well in [10,11], where some sufficient conditions for the total stability of a single fuzzy system have been introduced.

In this paper, the total stability of specific fuzzy control systems is studied. Specifically we consider the total stability of a feedback control system that consists of a linear plant and a fuzzy controller of Sugeno type, connected in series. In this direction, in section 2 we first show that this closed loop system may be reduced to an equivalent single fuzzy system of Sugeno type. Then, sufficient conditions for the total stability of the closed loop system will be introduced in section 3. Some examples in support of the theoretical results are also presented.

## 2. STRUCTURE OF THE FUZZY FEEDBACK CONTROL SYSTEM

Consider the system shown in Fig. 1, where the plant G is linear and described by

$$y(k+1)=\sum_{p=1}^{n}a_p y(k-p+1)+Ku(k) \tag{1}$$

*Received by the editors September 22, 1996 and in final revised form June 9, 1998
**Corresponding author

and the fuzzy controller is of Sugeno type whose i'th rule is defined by

$$\text{L}^i: If\, z_1(k)\ is\ F_1^i\ and \ldots and\ z_m(k)\ is\ F_m^i \quad \text{Then} \quad u^i(k) = b_0^i + \sum_{q=1}^{m} b_q^i e(k-q+1), \quad ,i=1,\ldots,l \qquad (2)$$

The fuzzy controller output, u(k), is inferred by considering the weighted average of the $u^i(k)$'s as follows

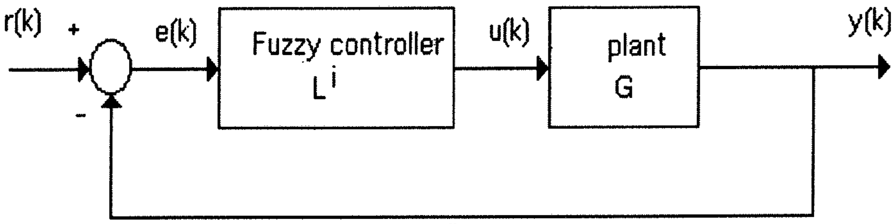



Fig. 1. The fuzzy feedback control system

$$u(k) = \frac{\sum_{i=1}^{l} w^i u^i(k)}{\sum_{i=1}^{l} w^i} \qquad (3)$$

where $\sum_{i=1}^{l} w^i > 0$, and the weight $w^i \geq 0,\ i=1,\ldots,l$ represents the overall truth value of the premise of the i'th implication for the controller input.

**Theorem 1:** The closed loop system shown in Fig. 1 and described by Eqs. (1) and (2) is equivalent to a fuzzy system of Sugeno type defined by

$$R^i: If\, z_1(k)\ is\ F_1^i\ and \ldots and\, z_m(k)\ is\ F_m^i \quad Then\ \ y^i(k+1) = \sum_{s=1}^{t} d_s^i y(k-s+1) + Kb_0^i + \sum_{q=1}^{m} Kb_q^i r(k-q+1) \qquad (4)$$

where

$$\begin{aligned} d_s^i &= a_s - Kb_s^i\ , \qquad i=1,\ldots,l \\ a_{n+j} &= 0,\ for\ j=1,\ldots,m-n\ and\ m>n \\ b_{m+j}^i &= 0,\ for\ j=1,\ldots,n-m\ and\ n>m\ and\ i=1,\ldots,l \\ t &= \max(n,m) \end{aligned} \qquad (5)$$

**Proof:** Based on (1) and (3), we have

$$y(k+1) = \sum_{p=1}^{n} a_p y(k-p+1) + K(\sum_{i=1}^{l} w^i u^i)/(\sum_{i=1}^{l} w^i) = \sum_{i=1}^{l} w^i [\sum_{p=1}^{n} a_p y(k-p+1) + Ku^i]/(\sum_{i=1}^{l} w^i) \qquad (6)$$

Substituting $u^i$ from (2) into (6), and by considering that *e(k)=r(k)-y(k),* we will have

$$y(k+1) = \sum_{i=1}^{l} w^i [\sum_{p=1}^{n} a_p y(k-p+1) + Kb_0^i + \sum_{q=1}^{m} Kb_q^i r(k-q+1) - \sum_{q=1}^{m} Kb_q^i y(k-q+1)]/(\sum_{i=1}^{l} w^i) \qquad (7)$$

Further simplification of (7) results in

$$y(k+1) = \sum_{i=1}^{l} w^i [\sum_{s=1}^{t} d_s^i y(k-s+1) + Kb_0^i + \sum_{q=1}^{m} Kb_q^i r(k-q+1)]/(\sum_{i=1}^{l} w^i) \qquad (8)$$

But, Eq. (8) represents the output of the system described by (4).

**Remark 1:** In the fuzzy system representation (4), $r(k)$ is the input; $y(k)$ is the output; $y(k)$, $y(k-1),\ldots,y(k-t+1)$ may be considered as the state of the system; $z_1(k),\ldots,z_m(k)$ are the premise variables which are functions of the states of the system.

## 3. STABILITY OF THE FUZZY CLOSED LOOP SYSTEM

In the stability analysis of the closed loop system, which is equivalently described by (4), the following cases may be considered:

**Case (i):** The total stability of the system (4) can be analyzed using the following theorem [10,11]. It may be noticed that by total stability we mean the following.

**Definition 1:** The system (4) is totally stable if and only if for any bounded input and any bounded initial state, the output and the state of the system are bounded.

**Remark 2:** It should be noticed that the total stability defined above is the one used in linear systems [9] which implies bounded input-bounded output stability for both linear and nonlinear systems. However, the concept defined by definition 1 is different from that which is usually adopted in nonlinear systems [12].

**Theorem 2**: The fuzzy system (4) is totally stable if there exists a common positive definite matrix P for all the subsystems of (4) such that

$$A_i^T P A_i - P < 0 \quad for \;\; i = 1,\ldots,l \tag{9}$$

where $A_i$ is defined as

$$A_i = \begin{bmatrix} d_1^i & d_2^i & . & . & . & d_{t-1}^i & d_t^i \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & & & . & . \\ . & . & . & & & . & . \\ . & . & . & & & . & . \\ 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 1 & 0 \end{bmatrix} \tag{10}$$

**Case(ii):** The asymptotical stability of the fuzzy system (4) may be considered for the case $b_0^i = 0,\ i = 1,\ldots,l,$ and $r \equiv 0$. In such case the system (4) is reduced to the following.

$$R^j: If\ z_1(k)\ is\ F_1^i\ and \ldots and\ z_m(k)\ is\ F_m^i\ Then\ y^i(k+1) = \sum_{s=1}^{t} d_s^i y(k-s+1) \qquad i = 1,\ldots,l \tag{11}$$

**Theorem 3[5]:** The equilibrium point of the fuzzy system (11) is globally asymptotically stable if there exists a common positive definite matrix $P$ for all the subsystems of (11) such that

$$A_i^T P A_i - P < 0 \qquad i = 1,\ldots,l \tag{12}$$

**Remark 3:** It may be noticed that the conditions for the asymptotical stability of the equilibrium point of the system (11) are the same as the conditions of the total stability of the system (4).

**Case(iii):**

**Corollary 1**: If $b_q^i = 0, q = 1,\ldots,m, i = 1,\ldots,l$ then (4) will be reduced to

$$If\ z_1(k)\ is\ F_1^i\ and \ldots and\ z_m(k)\ is\ F_m^i\ Then\ y^i(k+1) = \sum_{p=1}^{n} a_p y(k-p+1) + Kb_0^i \quad i = 1,\ldots,l \tag{13}$$

**Proof:** Substitution of the corollary conditions in (7), results in

$$y(k+1) = \sum_{i=1}^{l} w^i [\sum_{p=1}^{n} a_p y(k-p+1) + Kb_0^i] / (\sum_{i=1}^{l} w^i) \tag{14}$$

But, Eq. (14) represents the output of the system described by (13).

**Theorem 4:** The fuzzy system (13) is totally stable if the linear system (1) is totally stable.

**Proof:** Without loss of generality, we can assume that $m \leq n$. Then, from (13), we notice that $A_i = A, i = 1,\ldots,l$, where $A$ is defined as

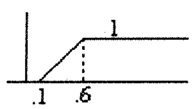
$$A = \begin{bmatrix} a_1 & a_2 & . & . & . & a_{n-1} & a_n \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & & & . & . \\ . & . & & . & & . & . \\ . & . & & & . & . & . \\ 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 1 & 0 \end{bmatrix} \tag{15}$$

Since the plant is totally stable, the zero state of the state description of (1) is asymptotically stable. Thus, based on the Lyapunov stability theorem [13] there exists a positive definite matrix $P$ such that $A^T PA - P < 0$, which implies $A_i^T PA_i - P < 0$ *for* $i = 1,\ldots,l$. Thus, the sufficient condition of Theorem 3 is satisfied. Therefore, the closed loop system (13) is totally stable.

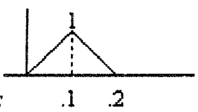
**Example 1**: Let us assume that the plant of Fig. 1 is described by

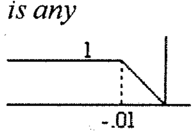
$$y(k+1) = 1.956y(k) - 0.960y(k-1) + .002u(k)$$

where the plant is stable and the fuzzy controller is specified by the following rules

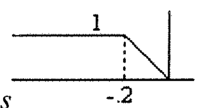
*If e(k) is* 

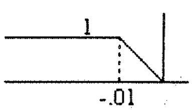
*and Δe(k) is any* *Then* $u^1(k) = 1.5$

*If e(k) is* 

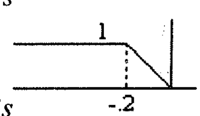
*and Δe(k) is* 

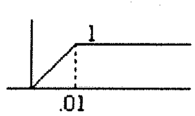
*Then* $u^2(k) = 0.8$

*If e(k) is* 

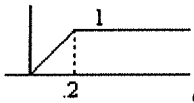
*and Δe(k) is* 

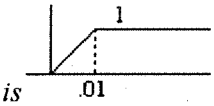
*Then* $u^3(k) = -1.5$

*If e(k) is* 

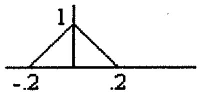
*and Δe(k) is* 

*Then* $u^4(k) = 1.1$

*If e(k) is* 

*and Δe(k) is* 

*Then* $u^5(k) = 1.4$

*If e(k) is* 

*and Δe(k) is any* *Then* $u^6(k) = 1$

Thus, the fuzzy controller satisfies the conditions of Corollary 1 and the plant satisfies the condition of Theorem 4. Furthermore, let the output scale factor of the controller be $SF_u = u_{ss}$ , where $u_{ss}$ is defined as

$$u_{ss} = \frac{1-\sum_{p=1}^{n} a_p}{K} \tag{16}$$

and is equal to 2 in this example. It may be noticed that based on relation (16), $u_{ss}$ is equal to $1/y_{ss}$ where $y_{ss}$ is the steady state unit step response of the plant (1). This causes the steady state unit step response of the closed loop system shown in Fig. 1 to be equal to one. Additionally, to have a proper performance for the step response of the system the error and change in error scale factors have been selected as, $SF_e = 10,\ \ SF_{\Delta e} = 1$. The step response of the fuzzy closed loop system is shown in Fig. 2.

**Example 2**: Let us assume that the plant description of Fig. 1 is changed to

$$y(k+1) = 1.795y(k) - 0.8y(k-1) + .01u(k)$$

and the rules of the fuzzy controller are the same as in Example 1, with $u_{ss} = 0.5, SF_u = u_{ss}, SF_e = 0.5, SF_{\Delta e} = 1$. In this case, the step response of the fuzzy closed loop system is shown in Fig. 3. It may be noted that though the plants are different for both systems of Examples 1 and 2, we have used almost the same controller.

**Case(iv):**

**Corollary 2:**If $b_q^i = b_q, q = 1,\dots,m, i = 1,\dots,l$,then (4) will be reduced to

$$R^i : If\ z_1(k)\ is\ F_1^i\ and...and\ z_m(k)\ is\ F_m^i\ \ Then\ \ y^i(k+1) = \sum_{s=1}^{t} d_s y(k-s+1) + Kb_0^i + \sum_{q=1}^{m} Kb_q r(k-q+1) \tag{17}$$

**Proof:** Substitution of the corollary conditions in (4) will result in (17) where $d_s^i = d_s,\ \ i = 1,\dots,l$

**Theorem 5**: The fuzzy system (17) is totally stable if there exists a positive definite matrix P such that $A_1^T P A_1 - P < 0$, Furthermore if $b_0^i = 0\ \ i = 1,\dots,l\ and\ \ r \equiv 0$, then the equilibrium point of fuzzy system (17) is globally asymptotically stable.

**Proof:** Based on Theorem 2, the system (17) is totally stable if there exists a common positive definite matrix $P$ for all subsystems such that $A_i^T P A_i - P < 0$. However, in this case, $A_i = A_1,\ i = 1,\dots,l$. Thus, this condition will be reduced to $A_1^T P A_1 - P < 0$. Furthermore, if $b_0^i = 0,\ i = 1,\dots,l\ and\ r \equiv 0$, the system (17) will be reduced to

$$If\ z_1(k)\ is\ F_1^i\ and...and\ z_m(k)\ is\ F_m^i\ \ Then\ \ y^i(k+1) = \sum_{s=1}^{t} d_s y(k-s+1)\quad i = 1,\dots,l \tag{18}$$

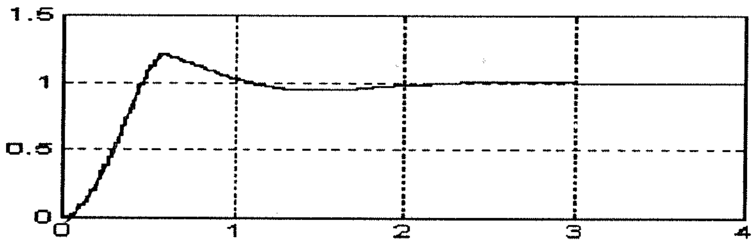



Fig. 2. The unit step response of the fuzzy closed loop system of example 1

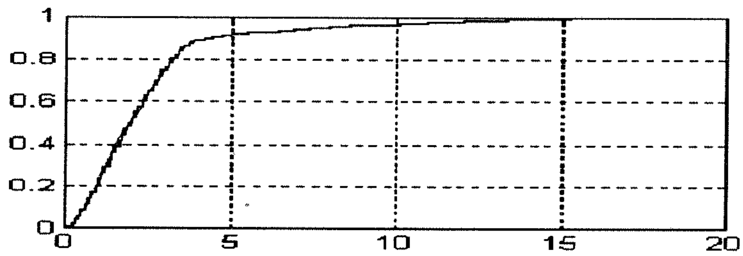


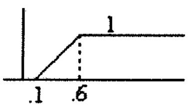

Fig. 3. The unit step response of the fuzzy closed loop system of example 2

Then, based on Theorem 3, we conclude that the equilibrium point of (18) is globally asymptotically stable.

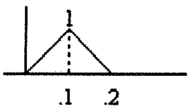
**Example 3:** Let us consider an unstable system described by

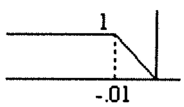
$$y(k+1) = 2.15y(k) - 1.2y(k-1) + .01u(k)$$

which is controlled by a fuzzy controller with the following rules

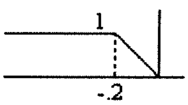
*If e(k) is* 

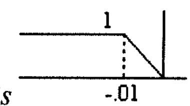
*and Δe(k) is any* *Then* $u^1(k) = 1.5 + 35.5e(k) - 40e(k-1)$

*If e(k) is* 

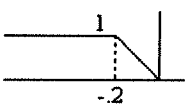
*and Δe(k) is* 

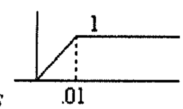
*Then* $u^2(k) = 0.8 + 35.5e(k) - 40e(k-1)$

*If e(k) is* 

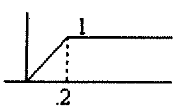
*and Δe(k) is* 

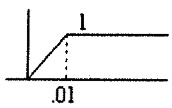
*Then* $u^3(k) = -1.5 + 35.5e(k) - 40e(k-1)$

*If e(k) is* 

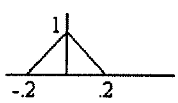
*and Δe(k) is* 

*Then* $u^4(k) = 1.1 + 35.5e(k) - 40e(k-1)$

*If e(k) is* 

*and Δe(k) is* 

*Then* $u^5(k) = 1.4 + 35.5e(k) - 40e(k-1)$

*If e(k) is* 

*and Δe(k) is any* *Then* $u^6(k) = 1 + 35.5e(k) - 40e(k-1)$

Furthermore, let the input-output scale factors be

$$SF_e = 0.5, \quad SF_{\Delta e} = 1, \quad SF_u = 0.5$$

Based on the system description, Eq. (5) and the coefficients of controller rules, matrix $A_1$ will be

$$A_1 = \begin{bmatrix} 1.795 & -0.8 \\ 1 & 0 \end{bmatrix}$$

Thus, by choosing the matrix $P$ as

$$p = \begin{bmatrix} 10 & -7.99 \\ -7.99 & 6.42 \end{bmatrix}$$

the expression $A_1^T P A_1 - P < 0$, as the condition of Theorem 5, is satisfied. Thus, the closed loop system is totally stable. Figure 4 represents the step response of the above fuzzy system. For comparison, the step response of the system when the controller is linear and described by

$$u(k) = 35.5e(k) - 40e(k-1)$$

is shown in Fig. 5. Both Figs. 4 and 5 indicate the stabilization ability of the fuzzy and the crisp controller respectively. However, although the consequence parts of the fuzzy controller rules are only different in $b_0^i$ compared to the crisp controller, the former has expressed much better performance.

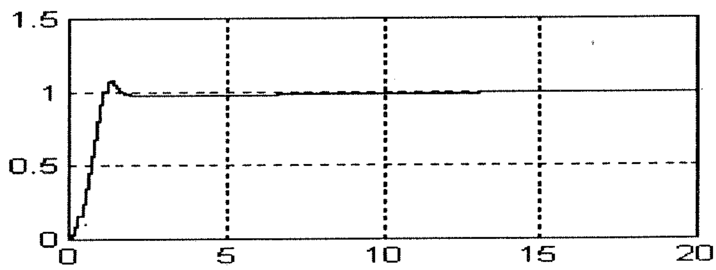



Fig. 4. The unit step response of the fuzzy closed loop system of example 3.

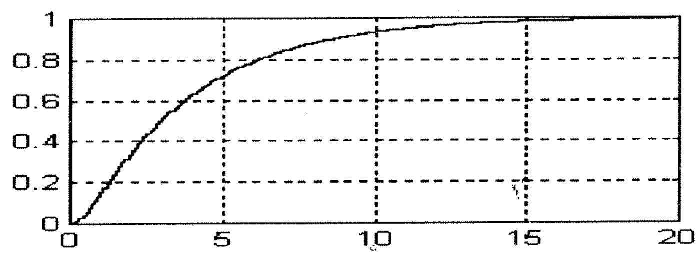



Fig. 5. The unit step response of the linear closed loop system of example 3.

## 4. CONCLUSION

In this paper, a further step toward the analytical stability study of fuzzy systems based on nonlinear system stability theorems has been taken. However, all poles linear systems have been considered as the controlled plants. In this direction, it would be useful to consider plants with other structures.

Furthermore, it would also be appropriate to study the approaches to fuzzy controller design based on the stability theorems of this paper.

*Acknowledgments:* The authors would like to thank the referees for their very helpful and constructive suggestions.

## REFERENCES


1. Lee, C. C., Fuzzy logic in control systems, Parts I and II, *IEEE Trans. SMC*, **20**, No. 2, (1990).
2. Driankov, D., Hellenddoorn, H., and Reinfrank, M., An introduction to fuzzy control. Springer-Verlag, (1993).
3. Schwartz, D. G. and Klir, G. J., Fuzzy logic flowers in Japan, *IEEE Spectrum* (1992).
4. Mendel, J. M., Fuzzy logic systems for engineering: A tutorial, *IEEE Proceeding*, **83**, No.3 (1995).
5. Tanaka, K. and Sugeno, M., Stability analysis and design of fuzzy control systems, *Fuzzy Sets and Systems*, **45**, No.2 (1992).
6. Wang, H. O., Tanaka, K. and Griffin, M.F., An approach to fuzzy control of nonlinear systems: Stability and design issues, *IEEE Trans. on Fuzzy Systems*, **4**, No.1 (1996).
7. Tanaka, K., Ikeda, T. and Wang, H. O., Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability, $H^{\infty}$ control theory, and linear matrix inequalities, *IEEE Trans. on Fuzzy Systems*, **4**, No.1 (1996).
8. Li, K. W., Turksen, I.B., Davison, E.J. and Smith, K.C., Stabilization of unstable and unintuitive plants by fuzzy control, *IEEE Trans. on SMC*, Part B, **27**, No.1 (1997).
9. Chen, C. T., Linear system theory and design, Holt, Rinehart and Winston (1990).
10. Tahani, V. and Sheikholeslam, F., Extension of new results on nonlinear systems stability to fuzzy systems, Proceedings, **2**, EUFIT'94, Second European Congress on Intelligént Techniques and Soft Computing, Aachen, Germany (1994).
11. Sheikholeslam, F., Stability analysis and design of nonlinear and fuzzy systems, *Ph.D Thesis,* Department of EECS Isfahan University of Technology, Isfahan, Iran (1998).
12. Slotine, J. J. E. and Li, W., Applied nonlinear control, Prentice-Hall (1991).
13. Vidyasagar, M., Nonlinear systems analysis, Prentice-Hall (1993).