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MANEUVERING TARGET TRACKING WITH HYBRID DATA MEASUREMENTS*

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Abstract– Most previous studies in estimation of the target position and velocity through Bearing Only Measurements (BOM) consider targets with constant velocity moving along a straight line. In this paper, state and measurement equations are presented for moving targets with constant acceleration by using the previously presented state vector in the Extended Modified Polar Coordinates (EMPC) system. In the BOM systems, by increasing the distance between target and observer (Own ship) the estimation accuracy of the target kinematic parameters degrades noticeably. In order to solve this problem, here the idea of hybrid data measurements is presented. In this approach both low rate range information, from active sensor, and high rate BOM are exploited. The improvement in the performance of the hybrid system compared to BOM system is represented through computer simulations.

Keywords– Bearing-only tracking, hybrid tracking, target motion analysis

1. INTRODUCTION

Nowadays passive systems applications, due to limitations of active systems, especially in surveillance applications have increased. In recent decades, the estimation problem of target kinematic parameters (position, velocity and acceleration) through bearing-only measurements (BOM), which is known as Target Motion Analysis (TMA) has been noticed [1-6]. In the bearing-only TMA, the target propagates either an electro-magnetic or an acoustic wave. Then the Directions of Arrival (DOA) of waves are measured by passive sensor(s) and these measurements are processed by tracking filters. Two types of tracking filters exist: i) processing a batch of data ii) processing the data recursively. In this study, the recursive filters are chosen. Extended Kalman Filter (EKF) has received considerable attention in recent years [7-9]. But previous studies have shown that the EKF in Cartesian coordinates exhibits unstable behavior characteristics when utilized for bearing-only TMA [10]. To solve this problem, target kinematic model is written in the so-called Modified Polar Coordinates (MPC) which leads to an EKF which is both stable and asymptotically unbiased [1].This model has been used in many researches on BOM-TMA [5], [11]. The MPC were originally conceived by K. R. Brown and significantly developed by H. D. Hoelzer and co-workers at IBM in the late 1970's [12]. Afterwards, in [1] state and measurement equations in MPC were derived for a constant velocity target moving along a straight line. Then State vector in MPC was extended to include target acceleration components in order to provide practical guidance for homing missiles with bearing only measurements [13]. We used this state vector in the Extended Modified Polar Coordinates (EMPC) system to derive exact state and measurement equations for maneuvering targets. In the BOM systems, by increasing the distance between the target and the observer (own ship) the estimation accuracy of the target kinematic parameters degrades noticeably. In order to solve this problem,

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the idea of hybrid data measurements is presented. In this idea both low rate range information, from active sensor, and high rate BOM are used. In section 2 the requirements on the observer and target paths is described. In section 3 state and measurement equations for a moving target with a constant acceleration are presented. In section 4 the idea of hybrid data measurements is explained. The performance of the BOM system and the hybrid system is compared in section 5 by computer simulations. Finally conclusions are presented in section 6.

2. PROBLEM FORMULATION AND ANALYSIS

In the problem of our interest, a target is moving with a constant acceleration in the x-y plane and the own ship, which is a single observer, is moving on the same x-y plane with a constant velocity on a circular path (see Fig.1). A basic principle of the position estimation through the BOM in the above scenario is that the observer motion dynamics must be one derivative higher than that of the target [2], [14], [15]. For example, a constant nonzero velocity observer (own ship) can estimate the position of a stationary target and an accelerating observer can estimate the position and velocity of a constant velocity target.

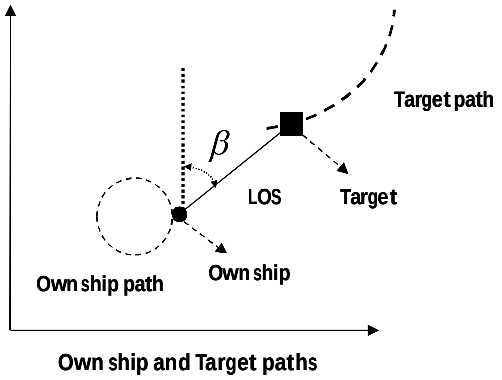

Fig.1. Own ship and target paths

A sensor on the own ship board collects N angular measurements T seconds apart. The Bearing angle, β, of the line connecting the sensor phase center to target is measured with respect to the y-axis:

$$z(kT) = \operatorname{atan}\frac{X_T(kT) - X_O(kT)}{Y_T(kT) - Y_O(kT)} + n(kT) \qquad k = 1,2,3,\ldots,N \tag{1}$$

where $z(kT)$ is the noisy angular measurement, ($X_T(kT)$, $Y_T(kT)$) are the unknown target coordinates at the kth time instant while ($X_O(kT)$, $Y_O(kT)$) are the known own ship coordinates. $n(kT)$ is the angular measurement noise and it is assumed to be a white Gaussian noise with zero mean and variance, σ^2_{angle} . To use both range information (active data) and angle information (passive data), an active sensor is considered on the own ship board which collects N' relative range of own ship-target measurements T' seconds apart where $T' \rangle T$ and $N' \langle N$.

$$r(mT') = \sqrt{[X_T(mT') - X_O(mT')]^2 + [Y_T(mT') - Y_O(mT')]^2} + \nu(mT') \qquad m = 1,2,3,\ldots,N' \tag{2}$$

where $r(mT')$ is the noisy relative range of the own ship-target measurement and $\nu(mT')$ is the range measurement noise which is assumed to be a white Gaussian noise with zero mean and variance, σ^2_{range}. Bearing angle measurements (in BOM systems) and range measurements (in hybrid- systems) are used in EKF to estimate the target kinematic parameters.

3. EKF IN BOM-TMA PROBLEM

Previous studies for BOM-TMA have shown that the EKF in Cartesian coordinates exhibits unstable behavior [10]. To solve this problem the target kinematic model is written in the MPC which leads to an EKF which is both stable and asymptotically unbiased [1]. In [1] it is shown that the use of MPC decouples the observable (the bearing angle) and unobservable (the target range) components of the state vector, and this decoupling prevents the ill conditioning of the covariance matrix of the estimation error which causes the filter instability. The Modified Polar (MP) state vector has been extended to EMPC system for moving target with a constant acceleration [13].
The state vector for EMPC system is:

$$\mathbf{y} = [y_1(t), y_2(t), y_3(t), y_4(t), y_5(t), y_6(t)]^T = \left[\dot{\beta}(\mathrm{t}), \dot{\mathrm{r}}(\mathrm{t})/\mathrm{r}(\mathrm{t}), \beta(\mathrm{t}), 1/r(t), \mathrm{a}_{\mathrm{T}\beta}(t), \mathrm{a}_{\mathrm{T}r}(t)\right]^T \quad (3)$$

where $\dot{\beta}(\mathrm{t})$ is the derivative of the measured angle $\beta(\mathrm{t})$, $\dot{r}/r$ is the relative range rate divided by relative range, $\beta(\mathrm{t})$ is the measured angle with respect to y-axis, $1/r$ is the inverse of relative range to own ship, $\mathrm{a}_{\mathrm{T}\beta}(t)$ is the target acceleration component perpendicular to the current Line Of Sight (LOS) and $\mathrm{a}_{\mathrm{T}r}(t)$ is the target acceleration component along the LOS. Based on the analysis reported in Appendix A, it has been shown that if the state vector for the EMPC system of a moving target with a constant acceleration is known at a given time t_0, the path of the target for $t \rangle t_0$ can be computed analytically as:

$$\mathbf{y}(t) = \mathbf{f}(\,\mathbf{y}(\mathrm{t}_0), \mathrm{t}, \mathrm{t}_0) \quad (4)$$

where $\mathbf{f}$ is a 6-dimentional vector function whose components are non-linear functions of the state vector at the initial time t_0 and the current time t. Equation (4) can be considered as the state equation in BOM-TMA problem. The measurement equation in BOM-TMA problem is given by:

$$\hat{\beta}(t) = \mathbf{H}\mathbf{y} + \mathrm{n}(\mathrm{t}) \quad (5)$$

Where

$$\mathbf{H} = [0\;0\;1\;0\;0\;0]$$

$\hat{\beta}(t)$ is the noisy measurement bearing angle and $n(t)$ is the zero-mean Gaussian white noise with variance $\sigma^2{}_{angle}$. It can be seen that in the EMPC system the state equation is non-linear and the measurement equation is linear. The discrete time version of Eq. (4) is generated by putting $t = kT$ and $t_0 = (k-1)T$, where T is the time period of the angle measurement and $k = 1,2,3,...$. The tracking filter in the EMPC system is obtained by applying the EKF to Eqs. (4) and (5) in discrete time form. The filtering equations are given by:

$$\begin{aligned}
&\mathbf{y}(0/0), \mathbf{P}(0/0)\\
&\mathbf{y}(\mathrm{k}/\mathrm{k}-1) = \mathbf{f}\,[\,\mathbf{y}(\mathrm{k}\text{-}1/\mathrm{k}-1), \mathrm{kT}, (\mathrm{k}\text{-}1)\mathrm{T}\,]\\
&\mathbf{A}_{\mathrm{y}}(k,k-1) = \frac{\partial \mathbf{f}[\mathbf{y}(\mathrm{k}\text{-}1/\mathrm{k}-1), \mathrm{kT}, (\mathrm{k}\text{-}1)\mathrm{T}]}{\partial \mathbf{y}(\mathrm{k}\text{-}1/\mathrm{k}-1)}\\
&\mathbf{P}(k/k-1) = \mathbf{A}_{\mathrm{y}}(k,k-1)\,\mathbf{P}(k-1/k-1)\,\mathbf{A}^T{}_{\mathrm{y}}(k,k-1) + \mathbf{Q}(k)\\
&\mathbf{H} = [001000]\\
&\mathbf{G}(k) = \mathbf{P}(k/k-1)\,\mathbf{H}^T\left[\mathbf{H}\mathbf{P}(k/k-1)\,\mathbf{H}^T + \sigma^2{}_{\mathrm{angle}}\right]^{-1}\\
&\mathbf{y}(\mathrm{k}/\mathrm{k}) = \mathbf{y}(\mathrm{k}/\mathrm{k}-1) + \mathbf{G}(k)\left[\hat{\beta}(k) - \mathbf{H}\mathbf{y}(\mathrm{k}/\mathrm{k}-1)\right]\\
&\mathbf{P}(k/k) = [\,\mathbf{I} - \mathbf{G}(k)\mathbf{H}\,]\;\mathbf{P}\,(k/k-1)\\
&k = 1,2,3,...
\end{aligned} \quad (6)$$

where $\mathbf{y}(k/k)$, $\mathbf{y}(k/k-1)$, $\mathbf{P}(k/k)$, $\mathbf{P}(k/k-1)$ are, respectively, the filtered state at time index k given all the measurements up to the time index k, the predicted state at time index k given all the measurements up to the time index $(k-1)$, the covariance matrix of estimation error ($\mathbf{y}(k)-\mathbf{y}(k/k)$) and the covariance matrix of estimation error ($\mathbf{y}(k)-\mathbf{y}(k/k-1)$). The values $\mathbf{y}(0/0)$ and $\mathbf{P}(0/0)$ are the initializations of the state estimate and the covariance matrix, respectively. In target kinematic parameters estimation, the estimation has a bias at long range. To avoid the bias a plant noise is considered in Eq. (4). $\mathbf{Q}(k)$ is the covariance matrix of plant noise at the kth time index. $\mathbf{G}(k)$ is the Kalman gain at the kth time index, $\mathbf{I}$ is the six-dimensional unit matrix. The above approach has been used in [1] and [11] for constant velocity target.

4. EKF IN HYBRID-TMA PROBLEM

Our simulations in bearing-only TMA show when the distance between target and own ship increases, because of lack of range information, the estimation accuracy of target kinematic parameters noticeably degrades. In order to solve the problem of the BOM system the practical idea that, as the measurement rate of active radar is low, the probability detection of radar decreases significantly is used. So using both low rate range information (from active radar) and high rate BOM (from passive radar) are proposed. This combination of active and passive data measurements is called hybrid data measurement. In this case when the active data (range measurement) is used, the state and measurement equations are changed as follow. State equation is similar to Eq. (4) but, in order to obtain the discrete time version of state equation it should be placing $t = kT'$ and $t_0 = (k-1)T'$ where T' is the time period of the range measurement by active radar and $k = 1,2,3,\ldots$. Measurement equation is changed to:

$$\begin{bmatrix} \hat{\beta}(t) \\ 1/\hat{r}(t) \end{bmatrix} = \mathbf{H}\mathbf{y} + \begin{bmatrix} \mathrm{n}(t) \\ v'(t) \end{bmatrix} \tag{7}$$

where $t = kT'$.

$\hat{\beta}(t)$ and $\frac{1}{\hat{r}(t)}$ are noisy bearing angle measurement and inverse of relative range measurement, respectively. $\begin{bmatrix} \mathrm{n}(t) \\ v'(t) \end{bmatrix}$ is the measurement noise vector. In this case $1/\hat{r}(t)$ is used instead of $\hat{r}(t)$ and its noise is shown by $v'(t)$. The range measurement noise is $v(t)$ and its variance is σ^2_{range}. The variance of $v'(t)$ is $\frac{\sigma^2_{\text{range}}}{r^4(t)}$. Here, $\mathbf{H}$ is a (2×6) matrix and is equal to $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Then the EKF equations are revised as:

$$\begin{aligned}
&\mathbf{y}(0/0), \mathbf{P}(0/0) \\
&\mathbf{y}(k/k-1) = \mathbf{f}\left[\mathbf{y}(k-1/k-1), kT', (k-1)T'\right] \\
&\mathbf{A}_{\mathbf{y}}(k,k-1) = \frac{\partial \mathbf{f}\left[\mathbf{y}(k-1/k-1), kT', (k-1)T'\right]}{\partial \mathbf{y}(k-1/k-1)} \\
&\mathbf{P}(k/k-1) = \mathbf{A}_{\mathbf{y}}(k,k-1)\mathbf{P}(k-1/k-1)\mathbf{A}^T{}_{\mathbf{y}}(k,k-1) + \mathbf{Q}(k) \\
&\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\
&\mathbf{G}(k) = \mathbf{P}(k/k-1)\mathbf{H}^T\left[\mathbf{H}\mathbf{P}(k/k-1)\mathbf{H}^T + \mathbf{C}\right]^{-1} \\
&\mathbf{y}(k/k) = \mathbf{y}(k/k-1) + \mathbf{G}(k)\left(\begin{pmatrix} \hat{\beta}(k) \\ \frac{1}{\hat{r}(k)} \end{pmatrix} - \mathbf{H}\mathbf{y}(k/k-1)\right) \\
&\mathbf{P}(k/k) = \left[\mathbf{I} - \mathbf{G}(k)\mathbf{H}\right]\mathbf{P}(k/k-1) \\
&k = 1,2,3,\ldots
\end{aligned} \tag{8}$$

where $\mathbf{C}$ is the covariance matrix of $[v(t) \quad v'(t)]^{T}$ as:

$$\mathbf{C}=\begin{bmatrix} \sigma^2{}_{\text{angle}} & 0 \\ 0 & \dfrac{\sigma^2{}_{\text{range}}}{r^4(t)} \end{bmatrix} \tag{9}$$

It can be seen that in this case, that compared to Eq. (6), the vector $\mathbf{H}$ is changed to a (2×6) matrix and the measurement scalar $\hat{\beta}(k)$ is changed to the measurement vector $\begin{pmatrix} \hat{\beta}(k) \\ \dfrac{1}{\hat{r}(k)} \end{pmatrix}$.

5. SIMULATION RESULTS

In this section three typical TMA scenarios are considered. In all scenarios a target is moving with a constant acceleration and own ship is moving on a circular path with a constant velocity. In the first scenario (section A) only the high rate bearing angle information for tracking the target (BOM tracking) is used, but in the second scenario (section B) in addition to using high rate bearing angle information, low rate range information (hybrid tracking) is also employed. In the third scenario (section C) a system that only uses the low rate range and bearing angle measurements for tracking the target is proposed. The sequence of angular measurements (for BOM tracking) and sequence of angular-range measurements (for hybrid tracking) are processed with EMPC-EKF and then the estimated target path in Cartesian coordinates is presented. At the end, the mean and Standard Deviation (STD) values of the tracking filter error by averaging the error along several independent trials of the same experiment are evaluated. In all scenarios the own ship is moving on a circular path whose center in Cartesian coordinates is $[0 \quad 0]^T$. The constant velocity of own ship is $50\ \text{m/s}$ and the overall angle of rotation is 4π radian (own ship moves on the circular path two times). The passive sensor collects $N=150$ angular measurements. The time period between consecutive measurements is $T=3\,\text{s}$. In the hybrid tracking scenario as the passive sensor collects $m=15$ angular measurements, the active sensor reports one range measurement so the period of active sensor measurement is $T'=45\,\text{s}$ (in this case the total range measurements is $N'=\frac{N}{m}=\frac{150}{15}$). The standard deviation of the angular measurement error is $\sigma_{\ angle}=1^{\circ}$ and the standard deviation of the range measurement error is $\sigma_{\text{range}}=\sqrt{50}\ \text{m}$. The target is initially located at the position of $[20000\,\text{m} \quad 15000\,\text{m}]^T$ its initial velocity vector is $[40\,\text{m}/s \quad 10\,\text{m/s}]^T$ and its constant acceleration vector is $[-0.1\,\text{m}/s^2 \quad 0.2\,\text{m}/s^2]^T$. In all scenarios the EMPC-EKF has been initialized on the basis of the first two angular and range measurements $z(1)$, $\text{r}(1)$, $\text{z}(2)$, $\text{r}(2)$.
thus:

$$\mathbf{y}(0/0)=\left[\frac{\text{z}(2)-z(1)}{T} \quad \frac{r(2)-r(1)}{T\,r(2)} \quad z(2) \quad \frac{1}{r(2)} \quad 0 \quad 0\right]^T$$

and initial covariance matrix is considered as:

$$\mathbf{P}(0/0) = diag\left[\frac{2\sigma^2 angle}{T^2},\ \frac{1}{T^2}\left[\left(\frac{1}{r(2)}\right)^2+\left(\frac{r(1)}{r^2(2)}\right)^2\right]\sigma^2 range,\ \sigma^2 angle,\ \frac{\sigma^2 range}{r^4(2)},\ \left(\frac{a_{T\beta}\max}{2}\right)^2,\ \left(\frac{a_{Tr}\max}{2}\right)^2\right]^T$$

where $a_{T\beta\,\max}$ and $a_{Tr\,\max}$ are the maximum expected of $a_{T\beta}$ and a_{Tr}, respectively.

Section A:

In this section the estimated and actual paths of the own ship and target, mean and STD of the relative range error and mean and STD of the bearing angle error are presented in Figs. 2 to 6, respectively.

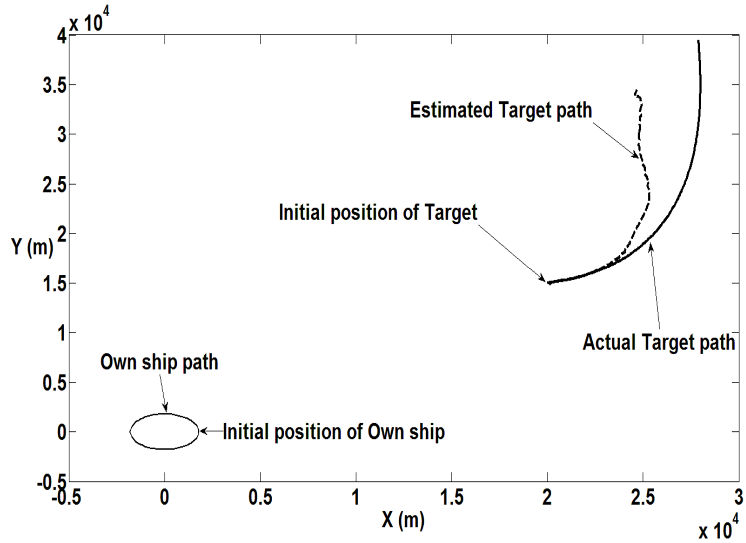

Fig. 2. Actual and estimated paths of the own ship and target of the BOM Tracking

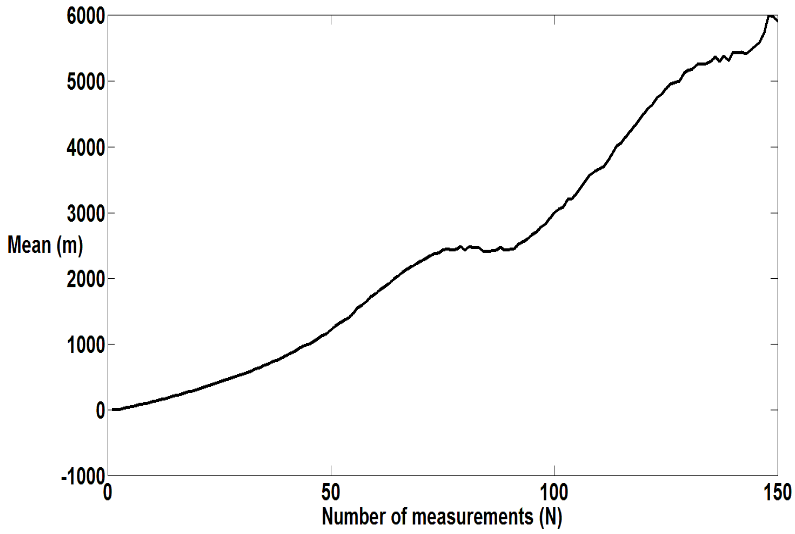

Fig. 3. Mean of the relative range error

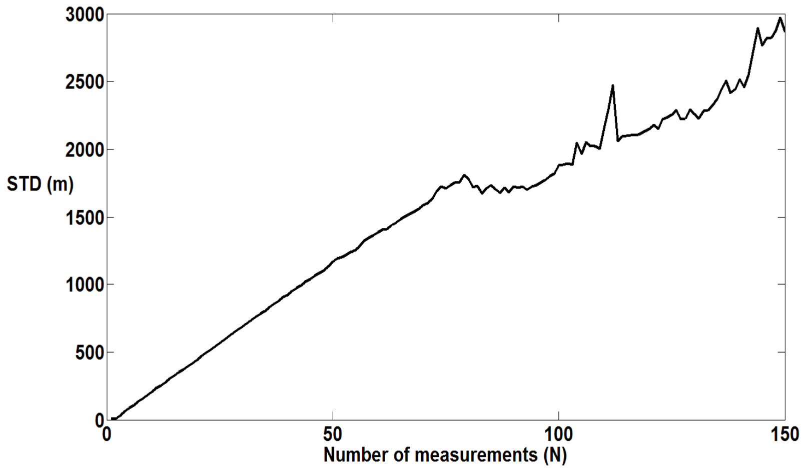

Fig. 4. STD of the relative range error

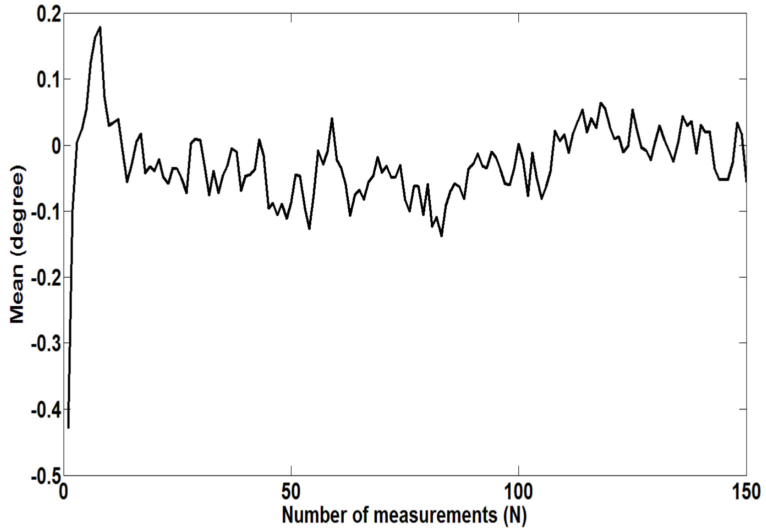

Fig. 5. Mean of the Bearing angle error

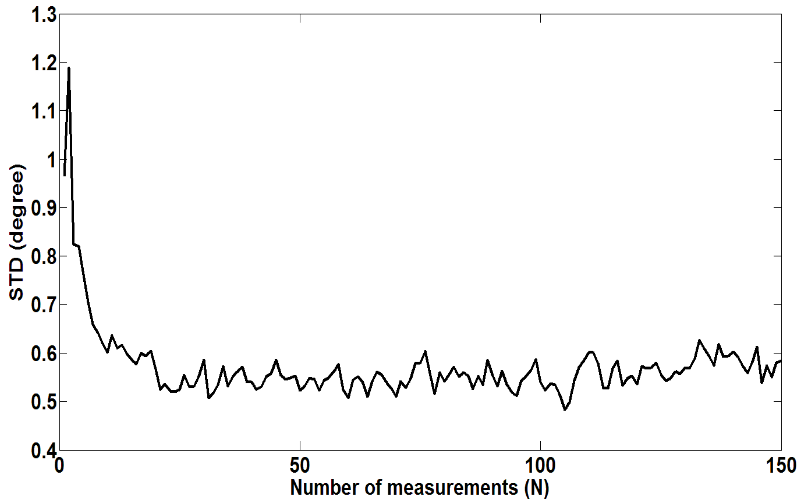

Fig. 6. STD of the Bearing angle error

Figures 3- 4 show that when the number of measurements increase or in other words the distance between the own ship and target increases then the mean and STD of relative range errors increase.

Section B:

In this section similar to section A the estimated and actual path of target, mean and STD of the relative range error and mean and STD of the bearing angle error for the second scenario are respectively presented in Figs. 7 to 11.

In Fig. 7 it can be seen that the estimated target path coincides with the actual path. From Fig. 8 it is completely clear when low rate range information (active data) is used, the relative range error is reduced. By comparing Figs. 8 and 3 it can be seen that at 150^{th} measurement the mean of relative range error for the hybrid tracking system is approximately 13 m, and for the BOM tracking system it is approximately 6000 m. It can be seen that in the hybrid tracking systems the estimation error of the target kinematic parameters compared to the BOM tracking systems has been considerably reduced. In all simulations it is assumed that the accurate positions of the own ship are known and no noise is considered for position parameters of the own ship.

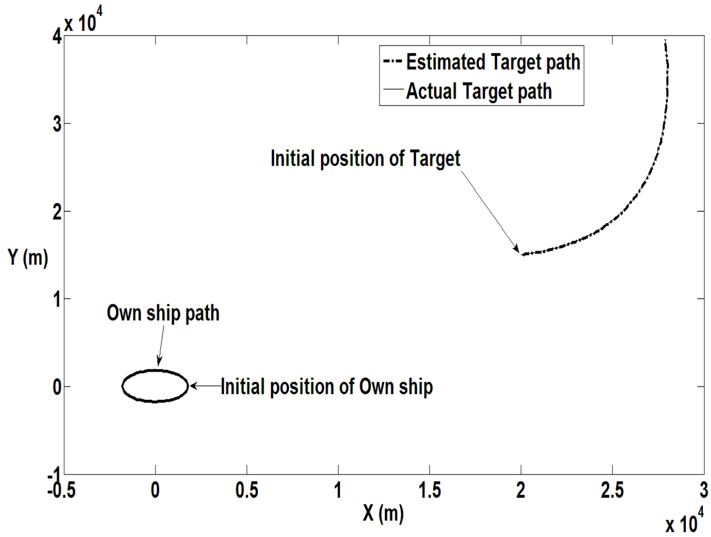

Fig. 7. Actual and estimated path of the own ship and target of the Hybrid Tracking

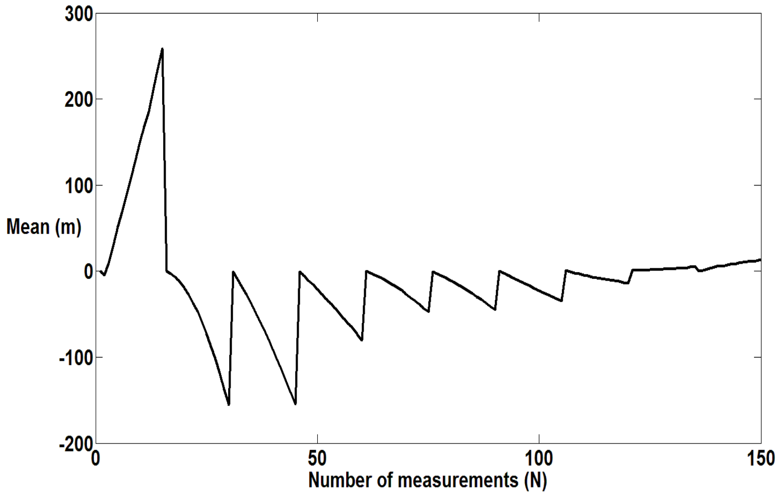

Fig. 8. Mean of the relative range error

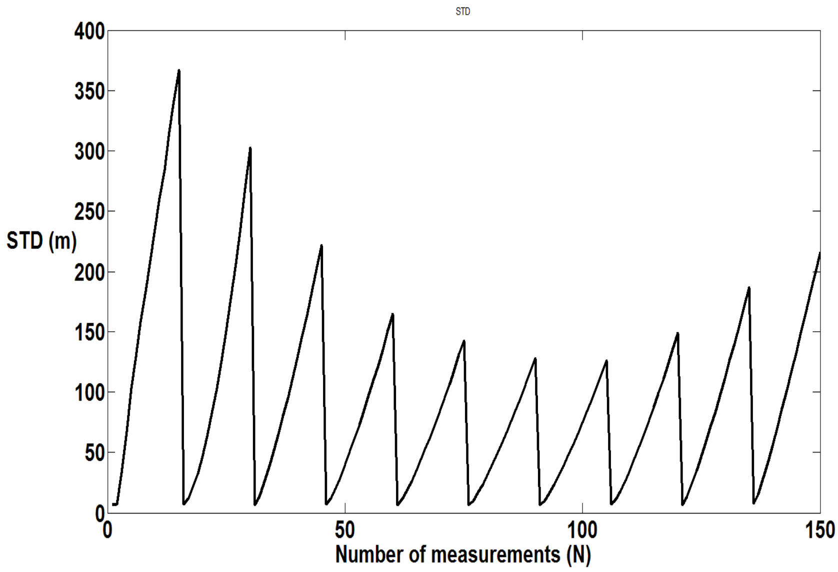

Fig. 9. STD of the relative range error

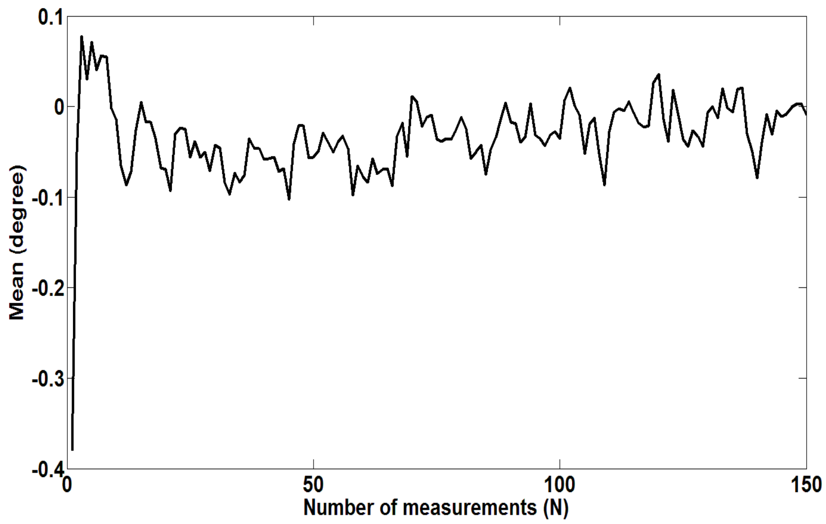

Fig. 10. Mean of the Bearing angle error

Section C:

In section B a hybrid system that combined the low rate range information with high rate bearing angle information in order to track the target is introduced. Now, a system that is only equipped with an active sensor, using low rate range and bearing angle measurements for tracking the target is proposed. This system here is called complete low rate system. Similar to the scenario proposed in section 5, an active sensor is considered that reports range and bearing angle measurements every 45 seconds. So the time period between consecutive measurements is $T' = 45\,s$ and the total measurement is 10. The mean and STD of the estimation error of the target kinematic parameters when the distance between the target and the own ship is 40 km are shown in tables 1 to 3 for BOM system, complete low rate system and hybrid system respectively.

Table 1. Mean and STD of estimation error of BOM system

Target components	x_T	y_T	V_{xT}	V_{yT}	a_{xT}	a_{yT}
Mean	960.5 m	859.9 m	2.78 m/s	4.23 m/s	-0.034 m/s^2	-0.019 m/s^2
STD	1876 m	1677 m	12.17 m/s	12.21 m/s	0.1 m/s^2	0.093 m/s^2

Table 2. Mean and STD of estimation error of complete low rate system

Target components	x_T	y_T	V_{xT}	V_{yT}	a_{xT}	a_{yT}
Mean	-6010 m	8856 m	-32.22 m/s	44.56 m/s	-0.06 m/s^2	0.13 m/s^2
STD	2430m	4828 m	11.72 m/s	18.69 m/s	1.017 m/s^2	1.02 m/s^2

Table 3. Mean and STD of estimation error of hybrid system

Target components	x_T	y_T	V_{xT}	V_{yT}	a_{xT}	a_{yT}
Mean	-15.7 m	-9.03 m	-1.52 m/s	-0.33 m/s	-0.006 m/s^2	-0.009 m/s^2
STD	257.5 m	289 m	3.24 m/s	4.71 m/s	0.019 m/s^2	0.038 m/s^2

By comparing the Tables 1 to 3 it is clear that the performances of the hybrid and BOM systems are better than that of the complete low rate system. The above results show that by combining high rate bearing angle information with low rate range information the estimation errors of the target kinematic parameters decrease significantly.

6. SUMMARY AND CONCLUSIONS

This paper consists of two main parts. In the first part, by using the extended state vector in EMPC system the state and measurement equations have been developed for the target that moves with a constant acceleration. In the second part the BOM system has been applied to the constant acceleration model. Computer simulations revealed that in BOM systems when the distance between the own ship and the target is low, the estimation accuracy of the target kinematic parameters is good. However, as the distance is increased the estimation accuracy of the target kinematic parameters significantly degrades. To solve this problem the idea of hybrid measurements has been employed. The corresponding simulations showed that by using hybrid measurements the performance of the tracking system considerably increases.

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APPENDIX A

Extended Modified Polar Coordinate (EMPC) Formulation of the BOM-TMA problem

In [1] the state and measurement equations have been derived for target moving with constant velocity. However, here, the state and measurement equations are derived for a moving target with constant acceleration. Consider the geometry depicted in Fig. 12, with the target and own ship confined to the same horizontal plane.

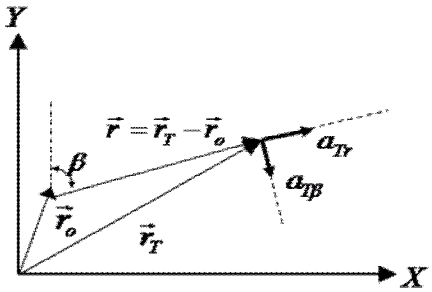

Fig. A1. Geometry of the own ship and target

The Cartesian state vector for this two-dimensional configuration is defined by

$$\mathbf{X}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}=\begin{bmatrix} r_x(t) \\ r_y(t) \\ V_x(t) \\ V_y(t) \\ a_{Tx}(t) \\ a_{Ty}(t) \end{bmatrix} \tag{A-1}$$

where

$$\begin{bmatrix} r_x(t) \\ r_y(t) \end{bmatrix}=\begin{bmatrix} r_{Tx}(t)-r_{ox}(t) \\ r_{Ty}(t)-r_{oy}(t) \end{bmatrix} \tag{A-2}$$

and

$$\begin{bmatrix} V_x(t) \\ V_y(t) \end{bmatrix}=\begin{bmatrix} V_{Tx}(t)-V_{ox}(t) \\ V_{Ty}(t)-\mathrm{V}_{oy}(t) \end{bmatrix} \tag{A-3}$$

denote the relative target position and velocity, respectively. A target which moves with a constant acceleration is considered. So $a_{Tx}(t)$ and $a_{Ty}(t)$ are constant values. In cartesian coordinate, the dynamic equations for a constant acceleration moving system are:

$$\begin{aligned} r_x(t)&=r_x(t_0)+(t-t_0)V_x(t_0)+\frac{1}{2}a_{Tx}(t-t_0)^2-\int_{t_0}^{t}(t-\tau)\,a_{ox}(\tau)d\tau \\ r_y(t)&=r_y(t_0)+(t-t_0)V_y(t_0)+\frac{1}{2}a_{Ty}(t-t_0)^2-\int_{t_0}^{t}(t-\tau)\,a_{oy}(\tau)d\tau \end{aligned} \tag{A-4}$$

where t_0 is the initial time.

The dynamic equations can be expressed in matrix notation. So we have

$$\mathbf{x}(t) = \mathbf{A}(t, t_0)\mathbf{x}(t_0) + \mathbf{u}(t, t_0) \tag{A-5}$$

where

$$\mathbf{A}(t,t_0) = \begin{bmatrix} 1 & 0 & (t-t_0) & 0 & \frac{(t-t_0)^2}{2} & 0 \\ 0 & 1 & 0 & (t-t_0) & 0 & \frac{(t-t_0)^2}{2} \\ 0 & 0 & 1 & 0 & (t-t_0) & 0 \\ 0 & 0 & 0 & 1 & 0 & (t-t_0) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A-6}$$

$$\mathbf{u}(t,t_0) = \begin{bmatrix} u_1(t,t_0) \\ u_2(t,t_0) \\ u_3(t,t_0) \\ u_4(t,t_0) \\ u_5(t,t_0) \\ u_6(t,t_0) \end{bmatrix} = \begin{bmatrix} -\int_{t_0}^{t} (t-\tau) a_{ox}(\tau) d\tau \\ -\int_{t_0}^{t} (t-\tau) a_{oy}(\tau) d\tau \\ -\int_{t_0}^{t} a_{ox}(\tau) d\tau \\ -\int_{t_0}^{t} a_{oy}(\tau) d\tau \\ 0 \\ 0 \end{bmatrix} \tag{A-7}$$

Equation (14) is considered as the Cartesian state equation. Again referring to the geometric configuration depicted in Fig. 12 the EMP state vector is defined by

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \\ y_6(t) \end{bmatrix} = \begin{bmatrix} \dot{\beta}(t) \\ \dot{r}(t)/r(t) \\ \beta(t) \\ 1/r(t) \\ a_{T\beta}(t) \\ a_{Tr(t)} \end{bmatrix} \tag{A-8}$$

where

$$r(t) = \sqrt{r^2{}_x(t) + r^2{}_y(t)} \tag{A-9}$$

$$\beta(t) = \tan^{-1}\left[\frac{\mathrm{r_x}(t)}{\mathrm{r_y}(t)}\right] \tag{A-10}$$

represent the relative range and bearing angle, respectively. In MPC system:

$$r_x(t) = r(t)\sin\beta(t)$$

$$r_y(t) = r(t)\cos\beta(t) \tag{A-11}$$

and

$$V_x(t) = \dot{r}(t)\sin\beta(t) + r(t)\dot{\beta}(t)\cos\beta(t)$$

$$V_y(t) = \dot{r}(t)\cos\beta(t) - r(t)\dot{\beta}(t)\sin\beta(t) \tag{A-12}$$

$$a_{xT}(t)=a_{Tr}(t)\sin\beta(t)+a_{T\beta}(t)\cos\beta(t)$$

$$a_{yT}(t)=a_{Tr}(t)\cos\beta(t)-a_{T\beta}(t)\sin\beta(t) \tag{A-13}$$

A one-to-one transformation which maps the EMP state vector into its Cartesian counterpart can now be deduced by combining (10) with (17), (20), (21), and (22). It yields

$$\mathbf{x}(t)=\mathrm{f}_x[\mathbf{y}(\mathbf{t})]=\frac{1}{y_4(t)}\begin{bmatrix}\sin y_3(t)\\ \cos y_3(t)\\ y_2(t)\sin y_3(t)+y_1(t)\cos y_3(t)\\ y_2(t)\cos y_3(t)-y_1(t)\sin y_3(t)\\ y_4(t)y_6(t)\sin \mathrm{y}_3(t)+y_4(t)y_5(t)\cos y_3(t)\\ -y_4(t)y_5(t)\sin \mathrm{y}_3(t)+y_4(t)y_6(t)\cos y_3(t)\end{bmatrix} \tag{A-14}$$

Accordingly, $\mathbf{x}(\mathbf{t_0})$ can be computed by letting $t=t_0$ and applying this transformation to the right-hand side of (14). After some manipulations this yields

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\\x_5(t)\\x_6(t)\end{bmatrix}=\frac{1}{y_4(t_0)}\begin{bmatrix}s_1(t,t_0)\cos y_3(t_0)+s_2(t,t_0)\sin y_3(t_0)\\ s_2(t,t_0)\cos y_3(t_0)-s_1(t,t_0)\sin y_3(t_0)\\ s_3(t,t_0)\cos y_3(t_0)+s_4(t,t_0)\sin y_3(t_0)\\ s_4(t,t_0)\cos y_3(t_0)-s_3(t,t_0)\sin y_3(t_0)\\ s_5(t,t_0)\cos y_3(t_0)+s_6(t,t_0)\sin y_3(t_0)\\ s_6(t,t_0)\cos y_3(t_0)-s_5(t,t_0)\sin y_3(t_0)\end{bmatrix} \tag{A-15}$$

Where

$$s_1(t,t_0)=Ty_1(t_0)+\frac{T^2}{2}y_4(t_0)y_5(t_0)+y_4(t_0)[u_1(t,t_0)\cos y_3(t_0)-u_2(t,t_0)\sin y_3(t_0)] \tag{A-16}$$

$$s_2(t,t_0)=1+Ty_2(t_0)+\frac{T^2}{2}y_4(t_0)y_6(t_0)+y_4(t_0)[u_1(t,t_0)\sin y_3(t_0)+u_2(t,t_0)\cos y_3(t_0)] \tag{A-17}$$

$$s_3(t,t_0)=y_1(t_0)+Ty_4(t_0)y_5(t_0)+y_4(t_0)[u_3(t,t_0)\cos y_3(t_0)-u_4(t,t_0)\sin y_3(t_0)] \tag{A-18}$$

$$s_4(t,t_0)=y_2(t_0)+Ty_4(t_0)y_6(t_0)+y_4(t_0)[u_3(t,t_0)\sin y_3(t_0)+u_4(t,t_0)\cos y_3(t_0)] \tag{A-19}$$

$$s_5(t,t_0)=y_4(t_0)y_5(t_0) \tag{A-20}$$

$$s_6(t,t_0)=y_4(t_0)y_6(t_0) \tag{A-21}$$

by differentiating (18) and (19) with respect to time and then combining the results with (10) and (22) it can lead to an inverse transformation which maps Cartesian states into EMP states. This transformation is

$$\mathbf{y}(t)=\mathbf{f}_y[\mathbf{x}(t)]=\begin{bmatrix}[x_3(t)x_2(t)-x_4(t)x_1(t)]\,/\left(x_1^2(t)+x_2^2(t)\right)\\ [x_3(t)x_5(t)+x_4(t)x_2(t)]\,/\left(x_1^2(t)+x_2^2(t)\right)\\ \tan^{-1}[x_1(t)/x_2(t)]\\ \dfrac{1}{\sqrt{x_1^2(t)+x_2^2(t)}}\\ [x_2(t)x_5(t)-x_1(t)x_6(t)]\Big/\sqrt{x_1^2(t)+x_2^2(t)}\\ [x_2(t)x_6(t)+x_1(t)x_5(t)]\Big/\sqrt{x_1^2(t)+x_2^2(t)}\end{bmatrix} \tag{A-22}$$

Substituting (24) into (31) yields

$$\mathbf{y}(t)=\mathbf{f}\left[\mathbf{y}(t_0), \mathrm{t}, \mathrm{t}_0\right]=\begin{bmatrix} \mathrm{f}_1[\mathbf{y}(t_0), \mathrm{t}, \mathrm{t}_0] \\ \mathrm{f}_2[\mathbf{y}(t_0), \mathrm{t}, \mathrm{t}_0] \\ \mathrm{f}_3[\mathbf{y}(t_0), \mathrm{t}, \mathrm{t}_0] \\ \mathrm{f}_4[\mathbf{y}(t_0), \mathrm{t}, \mathrm{t}_0] \\ \mathrm{f}_5[\mathbf{y}(t_0), \mathrm{t}, \mathrm{t}_0] \\ \mathrm{f}_6[\mathbf{y}(t_0), \mathrm{t}, \mathrm{t}_0] \end{bmatrix}=\begin{bmatrix} \left[\, s_3(t,t_0)s_2(t,t_0)-s_4(t,t_0)s_1(t,t_0)\right] \big/ \left[s_1^{\,2}(t,t_0)+s_2^{\,2}(t,t_0)\right] \\ \left[\, s_3(t,t_0)s_1(t,t_0)+s_4(t,t_0)S_2(t,t_0)\right] \big/ \left[s_1^{\,2}(t,t_0)+s_2^{\,2}(t,t_0)\right] \\ y_3(t_0)+\tan^{-1}\left[\, s_1(t,t_0)/\, s_2(t,t_0)\right] \\ y_4(t_0)\Big/\sqrt{s_1^{\,2}(t,t_0)+s_2^{\,2}(t,t_0)} \\ \dfrac{1}{y_4(t_0)}\left(\left[s_2(t,t_0)s_5(t,t_0)-s_1(t,t_0)s_6(t,t_0)\right]\Big/\sqrt{s_1^{\,2}(t,t_0)+s_2^{\,2}(t,t_0)}\right) \\ \dfrac{1}{y_4(t_0)}\left(\left[s_2(t,t_0)s_6(t,t_0)+s_1(t,t_0)s_5(t,t_0)\right]\Big/\sqrt{s_1^{\,2}(t,t_0)+s_2^{\,2}(t,t_0)}\right) \end{bmatrix} \quad \text{(A-23)}$$

By Eq. (32) if the EMP of the target is known at a given time t_0, then the EMP of the target can be calculated for $t \rangle t_0$.Since target bearing is a component of the EMP state vector, the measurement equation for BOM-TMA is expressed in the simple linear form

$$\hat{\beta}(t)=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}\mathbf{y}(t)+\mathrm{n}(\mathrm{t}) \quad \text{(A-24)}$$

where $n(t)$ is defined in section 3.

Although the preceding results are expressed in continuous form, discrete time equations of state and measurement forms are obtained by considering $t = kT$, $t_0 = (k-1)T$, where $k = 1,2,3,...$ and T is constant sampling period.